ISRAEL JOURNAL OF MATHEMATICS 89 (1995), 1–59

THE CONTINUOUS SPECTRUM OF THE RELATIVE TRACE FORMULA FOR GL(3) OVER A QUADRATIC EXTENSION

BY

HERVÉ JACQUET*

Department of Mathematics, Columbia University
New York, NY 10027, USA

ABSTRACT

Let E/F be a quadratic extension of number fields, G the group $\mathrm{GL}(3,E)$ regarded as an algebraic group over F and U a quasi-split unitary group in three variables. Let also θ be a generic character of a maximal unipotent subgroup N of G. We derive an explicit expression for the integral

$$\int\int K_{\mathrm{cont}}(u,n)du\theta(n)dn$$

where K_{cont} is the continuous part of the kernel attached to a smooth function of compact support on $G(\mathbb{A})$. In particular, we prove that this expression is absolutely convergent. The result can be used to show that a cuspidal representation of G contains a vector ϕ such that $\int \phi(u)du \neq 0$ if and only if it is a base change from a representation of $\mathrm{GL}(3,F)$.

1. Introduction

Let E/F be a quadratic extension of number fields. We denote by σ or $z \mapsto \overline{z}$ the Galois conjugation and by x^* the conjugate transpose of a matrix with entries in E. Let G be the group $\mathrm{GL}(n,E)$ regarded as an algebraic group over E and $S \subset G(E)$ the variety of invertible Hermitian matrices:

$$S = \{s \in G(E)\colon s^* = s\}. \tag{1}$$

* Partially supported by NSF Grant DMS-91-01637.

Received October 12, 1993 and in revised form March 24, 1994

The group $G(E)$ operates on S by

$$(g,s) \mapsto g^* s g.$$

If v is a finite place of E, we let R_v be the ring of integers in E_v and set $K_v = \mathrm{GL}(n, R_v)$. If v is a real place, we let K_v be the orthogonal group $O(n,\mathbb{R})$; if v is a complex place, we let K_v be the unitary group $U(n,\mathbb{C})$. We set $K = \prod K_v$. The notation $\|g\|$ has the usual meaning of a norm function on the group $G(E_{\mathbb{A}})$. We let

$$G^1 = \{g \in G(E_{\mathbb{A}}) \colon |\det(g)| = 1\}. \tag{2}$$

In general, we follow the notations of [A1] and [A2].

Suppose that Φ is a smooth function of compact support on $S(F_{\mathbb{A}})$. We will only consider the case of a decomposable function

$$\Phi(s) = \prod_v \Phi_v(s_v), \tag{3}$$

where, for almost all finite places v, the function Φ_v is the characteristic function, of $S_v \cap K_u$ if v is an inert place under the place u of E, and of $S_v \cap (K_{u_1} \times K_{u_2})$ if v splits into u_1, u_2. Note that Φ is the restriction to $S(F_{\mathbb{A}})$ of a smooth function of compact support on $G(E_{\mathbb{A}})$. We set

$$K_\Phi(g) = \sum_{\xi \in S(F)} \Phi(g^* \xi g). \tag{4}$$

LEMMA 1.1: *The sum (4) converges absolutely (and the resulting function is invariant on the left under $G(E)$). Furthermore, there is $c > 0$ and $N > 0$ such that*

$$|K_\Phi(g)| \le c\|g\|^N.$$

Finally, there is a finite set T of $F^\times$ and a finite subset X of $S(F)$ such that

$$\Phi(g^* \xi g) \neq 0$$

implies $\det \xi \in T N_{E/F}(E^\times)$ and ξ belongs to the orbit under $G(E)$ of some element of X.

Proof: Since Φ has compact support, for g in a compact set, the support of the sum is finite. This proves the first assertion. We pass to the second assertion.

Recall Φ is the restriction of a smooth function of compact support f on the group $G(E_{\mathbb{A}})$. It is well known that, for suitable c and N,

$$\sum_{\gamma\in G(E)} |f(x^{-1}\gamma y)| \le c\|y\|^N.$$

Since

$$|K_\Phi(g)| \le \sum_{\gamma\in G(E)} |f(g^*\gamma g)|$$

the conclusion follows.

For the third assertion, we remark that $\Phi(g^*\xi g) \neq 0$ implies that there is a finite set X of places such that for all $v \notin X$ the product $\det g_v\overline{g}_v \det\xi$ is a unit in E_v. Thus $\det\xi$ is a norm at places not in X. Since the only invariant of Hermitian matrices at a finite place is the class of the determinant modulo the group of norms, the class of ξ is uniquely determined at all places not in X and our assertion follows. ∎

Our long term goal is to define the projection $K_{\Phi,\mathrm{cusp}}$ of a function K_Φ on the space of cusp-forms and to identify the projection space with the space of cusp forms on $\mathrm{GL}(n,E)$ which are base change of cusp-forms on $\mathrm{GL}(n,F)$. In more detail, it is easily seen that if a cuspidal representation π contains a form ϕ whose integral against a function K_Φ is non-zero, then there exists a unitary group U and a form ϕ in the space of π such that

$$\int_{U(F)\backslash U(F_{\mathbb{A}})} \phi(u)du \neq 0. \tag{5}$$

We say then that π is distinguished with respect to U. The integral defines then a non-zero linear form on the space of π which invariant under the group $U(F_{\mathbb{A}})$. At a place v of F which splits into v_1 and v_2 in E, the groups $\mathrm{GL}(n,E_{v_1})$, $\mathrm{GL}(n,E_{v_2})$ are isomorphic to $\mathrm{GL}(n,F_v)$ and U_v can be identified with the twisted diagonal subgroup $\{(g,{}^t g^{-1})\}$ in the product. Thus the existence of the linear form implies that the representations $g\mapsto\pi_{v_1}(g)$ and $g\mapsto\pi_{v_2}({}^tg^{-1})$ are contragredient to one another, or, equivalently, that $\pi_{v_1}\simeq\pi_{v_2}$. If, on the contrary, v if finite and inert in E under the place u of E and π_u is unramified, then π_u is invariant under Galois conjugation. It follows that the representations $g\mapsto\pi(g)$ and $g\mapsto\pi(\overline{g})$ are equivalent at almost all places and thus are equivalent. Hence π must be a quadratic base change by [AC]. Our ultimate goal is to prove that conversely a quadratic base change is distinguished with respect to some unitary group. Of course for $n = 1$, a distinguished representation is an idèle-class character which is trivial on the group of elements of norm 1. Such a character is a quadratic base change, that is, of the from $z \mapsto \chi(z\bar{z})$, where χ is an idèle-class character of F. The problem at hand takes its origin in the reference [HLR]. In particular the above argument on the Galois invariance of a distinguished representation can be found there. The dual case, where roughly speaking the role of the groups U and $\mathrm{GL}(n, F)$ are exchanged, is discussed at length in [yF1].

In order to prove the conjecture, one must obtain an explicit expression for the difference

$$K_\Phi(g) - K_{\Phi,\mathrm{cusp}}(g).$$

Part of the difficulty is to establish that the final formula is **absolutely convergent**.

For the time being n is arbitrary (however, for an even n we would have to introduce the group of unitary similitudes). Our plan of attack is as follows. We will fix a finite subset X of $S(F)$ and consider only functions Φ such that $\Phi(g^*\xi g) \neq 0$ implies ξ is in the orbit of some point of X under $G(E)$. Let U_ξ be the unitary group which fixes $\xi \in X$. Then there are compactly supported functions $f_\xi, \xi \in X$, on $\mathrm{GL}(n, E_\mathbb{A})$ such that

$$\Phi(g^*\xi g) = \int_{U_\xi(F)\backslash U_\xi(F_\mathbb{A})} f_\xi(ug)du \tag{6}$$

and

$$\Phi(s) = 0,$$

if s is not in the orbit under $G(E_\mathbb{A})$ of a point of X. If we set, as usual,

$$K_f(x, y) = \sum_{\gamma\in\mathrm{GL}(n,E)} f(x^{-1}\gamma y) \tag{7}$$

then we see that

$$K_\Phi(g) = \sum_{\xi\in X} \int_{U_\xi(F)\backslash U_\xi(F_\mathbb{A})} K_{f_\xi}(u, g)du.$$

In fact, we will choose an idèle-class character ω of E which is itself a quadratic base change. We will set:

$$K_{\Phi,\omega}(g) = \int_{E_\mathbb{A}^\times/E^\times} \omega(z)K_\Phi(gz)d^\times z = \int_{E_\mathbb{A}^\times/E^\times} \omega(z) \sum_{\sigma\in S(F)} \Phi(g^* z\sigma g)d^\times z, \tag{8}$$

$$K_{f,\omega}(x,y) = \int \omega(z) \sum f(x^{-1}\gamma z y) d^\times z. \tag{9}$$

Then

$$K_{\Phi,\omega}(g) = \sum_\xi \int_{U_\xi(F)\backslash U_\xi(F_\mathbb{A})} K_{f_\xi,\omega}(u,g) du. \tag{10}$$

Our next step will be to consider the spectral expansion of each kernel

$$K_{f_\xi,\omega}(x,y) = \sum_\chi K_{f_\xi,\chi}(x,y),$$

the sum over all cuspidal data χ. We recall that if $P = MN$ is a parabolic subgroup of G then we write $M(E_\mathbb{A}) = A_P M^1$ where A_P is the split component of M and M^1 the group of $m \in M(E_\mathbb{A})$ such that $|\mu(m)| = 1$ for every rational character μ of M. Suppose π is a cuspidal automorphic representation of M^1. Whenever convenient we will identify such a representation to a representation of $M(E_\mathbb{A})$ trivial on A_P. Two triples (P, M, π) and (P', M', π') are equivalent if there is $w \in G(E)$ such that w conjugates M to M' and transforms π into π'. A cuspidal datum is an equivalence class for this relation. At this point we would like to set

$$K_{\Phi,\chi}(g) = \sum_\xi \int_{U_\xi(F)\backslash U_\xi(F_\mathbb{A})} K_{f_\xi,\chi}(u,g) du, \tag{11}$$

so as to have

$$K_{\Phi,\omega}(g) = \sum_\chi K_{\Phi,\chi}(g).$$

The cuspidal part of $K_{\Phi,\omega}$ is then defined as the sum of the $K_{\Phi,\chi}$, where χ is a cuspidal representation of G^1. If χ is not a cuspidal representation of G^1, then one wants to obtain an explicit expression for $K_{\Phi,\chi}(g)$ of roughly the following form:

$$\sum_\xi \sum_P \int \sum_\phi \Xi(I_P(\pi, f_\xi)\phi, \pi) \overline{E(g,\phi,\pi)}\, d\pi.$$

Here the second sum is over all standard parabolic subgroups $P = MN$; for each such P, $d\pi$ denotes a measure supported by the representations of $M(E_\mathbb{A})$ whose restriction to M^1 are in the discrete part of the spectrum determined by χ. The linear form

$$\phi \mapsto \Xi(\phi, \pi)$$

on the space of the representation $I_P(\pi)$ induced by π is invariant under U_ξ. Thus the representations induced by those representations π which support the measure $d\pi$ should be invariant under the Galois group. The inner sum is over an orthonormal basis of the induced representation and $E(g,\phi,\pi)$ is an Eisenstein series.

In this paper, we will consider the case $n=3$. Furthermore, we will let θ be a generic character of the maximal unipotent subgroup N_0 of G and compute only the difference

$$(12) \qquad \int_{N_0(F)\backslash N_0(F_{\mathbb{A}})} K_\Phi(n)\overline{\theta(n)}dn - \int_{N_0(F)\backslash N_0(F_{\mathbb{A}})} K_{\Phi,\mathrm{cusp}}(n)\overline{\theta(n)}dn.$$

Our aim will be to obtain for such an expression a formula of roughly the following form:

$$(13) \qquad \sum_\xi \sum_P \int \sum_\phi \Xi(I_P(\pi,f_\xi)\phi,\pi)\overline{\int E(n,\phi,\pi)\theta(n)dn}\, d\pi.$$

As before, we sum over all standard parabolic subgroups $P=MN$. For a given ξ and a given P, the integral is with respect to a certain measure on the set of **cuspidal** automorphic representations π of $M(E_{\mathbb{A}})$. As before, $\phi \mapsto \Xi(\phi,\pi)$ is a certain linear form on the induced representation $I_P(\pi)$ which is invariant under U_ξ and the representations induced by those π which support the measure $d\pi$ are actually invariant under Galois conjugation. When P is of type $(2,1)$ then $M \simeq \mathrm{GL}(2)\times\mathrm{GL}(1)$ and $\pi=\pi_1\otimes\pi_2$ where π_1 and π_2 are invariant under Galois conjugation and thus quadratic base change. When P is of type $(1,1,1)$ then $\pi=\pi_1\otimes\pi_2\otimes\pi_3$ and either each π_i is a quadratic base change or two of the characters are exchanged by Galois conjugation and the third one is a quadratic base change. Of course, the two cases are not exclusive of one another.

An important feature of this formula is its **absolute convergence**. The exact nature of the linear form Ξ is in a sense irrelevant. Nonetheless, it is worth noting that in the case of a parabolic subgroup of type $(2,1)$ the formula for the linear form Ξ is based on the fact that the inducing representation π_1 is distinguished for some form of the unitary group; even if we restrict our attention to the case where U_ξ is quasi-split, all unitary subgroups of the group $\mathrm{GL}(2)$ appear implicitly in the formula for Ξ.

Together with the ongoing work of Ye, the above formula will be enough to establish the conjecture for $n=3$ (under some restrictions on the quadratic extension). See [JY] and [Y]. The case $n=2$ was treated in [Y1], mainly from a formal point of view, in the sense that some of the analysis was insufficient.

The techniques of the present paper can be used to make the discussion rigorous. Similarly in [J1], [J2], it was implicitly assumed that the ground field was $\mathbb{Q}$ and the extension E an imaginary extension of $\mathbb{Q}$. The difficulty was there are infinitely many idèle-class characters which are unramified at all places; this introduces infinite sums, the convergence of which has to be established. The estimates sketched in the third section can be used to make the discussion complete. We hope the present paper does not contain essential errors. Since the general case is very difficult and would entail the use of more elaborate technics as in the work of Arthur, we feel it is reasonable to limit the scope of this paper to the case $n = 3$.

A similar analysis is attempted in [yF2] and [yF3] for different, more elaborate, situations. The reference [yF2] treats the "dual case" in the context of $\mathrm{GL}(n)$. While the paper is suggestive, it contains a very large number of serious errors and omissions. The reference [yF3] is also flawed to a substantial extent.

The material is arranged as follows. We review some properties of the truncation operator in Section 2. In particular, we explain how the truncation operator is used to prove the formula above. In Section 3, we show that the Eisenstein series induced from the parabolic subgroup of type $(1,1,1)$ can be majorized on the "imaginary plane." The result, Proposition 3.5, can be taken for granted at first reading. The proof uses standard techniques. Section 4 is an auxiliary Section where we estimate two infinite sums (Lemma 4.1 and 4.2). Again the result, which is elementary anyway, can be taken for granted at first reading. The result is used in Section 5: we study Arthur's second formula for the truncation of an Eisenstein series (induced from cusp forms). Replacing each term by its absolute value, we obtain a series of positive terms, in a suitable domain; we estimate the series of positive terms (Proposition 5.1) and show that it is integrable over the quasi-split unitary group. The crucial section is Section 6: we consider Arthur's second formula for a truncated Eisenstein series induced from a parabolic P and compute the integral of the truncated Eisenstein series over the quasi-split unitary group U. The integral can be written as a sum of terms, indexed by the double cosets of $U\backslash G/Q$, where Q is any associate of P. Because of the cuspidality of the datum, only certain double cosets can contribute a nonzero term. The computations in this Section are purely formal in case $n > 3$, but in case $n = 3$, they are justified by the estimates of Section 5. See the concluding remark of Section 6 for attributions. Finally, formulas for the integral of a truncated Eisenstein series are obtained in Section 7. They are really the core of this paper. The most difficult case is the case of the minimal parabolic subgroup. The relevant formula is obtained in Proposition 7.3. The formula is reminiscent of Arthur–Langlands formula for the scalar product of two truncated Eisenstein series. Finally formula (13) is obtained in Section 8. A subtle feature of this formula is the cancellation of the singularities. The linear form $\Xi(.,\pi)$ is a meromorphic function of π; it may have a pole on a line which intersects the range of integration. However the Eisenstein series has a zero on such a line and the product which appears in (13) is actually holomorphic on the range of integration. A similar cancellation appears in the work of Ye ([Y1]) and reappears in the references [yF]. Also, we prove the formula only when X is reduced to one point ξ such that the group U_ξ is quasi-split and the corresponding function f_ξ is a convolution product of two K-finite functions and for one specific quasi-split unitary group. This will suffice for the application we have in mind.

2. A review of truncation

For simplicity we fix $\xi \in S(F)$ so that $X = \{\xi\}$; we assume the group $U = U_\xi$ is quasi-split. We also fix the character ω; we assume it is a base change and, as usual, we assume that it is trivial on the split component of GL(1). We write f for f_ξ. We assume that f is K-finite. We write K_G for $K_{f,\omega}$. Instead of K_Φ we simply consider

$$\int_{U(F)\backslash U(F_{\mathbb{A}})} K_G(u,g)du.$$

For the convenience of the reader, we extract some simple facts from the work of Arthur. We follow his notations (A *trace formula for reductive groups I and II*, [A1], [A2]).

PROPOSITION 2.1: *Suppose Ω_0 is a compact set of G^1. Then, if T is sufficiently regular, for all $x \in \Omega_0$ and all y in $G(E_{\mathbb{A}})$:*

$$K_G(x,y) = \Lambda_2^T K_G(x,y),$$

where $\Lambda_2^T K_G(x,y)$ denotes truncation with respect to the second variable; similarly,

$$K_{G,\chi}(x,y) = \Lambda_2^T K_{G,\chi}(x,y),$$

for all $x \in \Omega_0$, all $y \in G(E_{\mathbb{A}})$ and all cuspidal data χ. Furthermore, given a Siegel set $\mathfrak{S}$ in G^1 and an integer N, there is a c such that for $x \in \Omega_0$ and $y \in \mathfrak{S}$,

$$\sum_{\chi} |K_\chi(x,y)| \leq c\|y\|^{-N}.$$

Proof: Recall the definition of the truncation operator:

$$\Lambda^T\phi(y) = \sum_P (-1)^{\dim(A/Z)} \sum_{P(E)\backslash G(E)} \hat{\tau}_P(H(\delta y) - T) \int_{N(F)\backslash N(\mathbb{A})} \phi(n\delta y)dn.$$

Thus

$$\Lambda_2^T K_G(x,y) =$$

$$\sum_P (-1)^{\dim(A/Z)} \sum_{\delta \in P(E)\backslash G(E)} \hat{\tau}_P(H(\delta y) - T) \int_{N(F)\backslash N(\mathbb{A})} K_G(x, n\delta y)dn.$$

Now

$$\int K_G(x,ny)dn = \sum_{\gamma\in P(E)\backslash G(E)} K_P(\gamma x, y).$$

Next, if $K_P(\gamma x, my) \neq 0$ for some $m \in M^1$, then we have

$$nmy \in \gamma x \Omega.$$

for some $n \in N(E_{\mathbb{A}})$; here Ω is a fixed compact set of G^1, depending only on the support of f. It follows that there is T_0 such that

$$\hat{\tau}_P(H(\gamma x) - H(y) - T_0) = 1.$$

The parameter T_0 depend only on the support of f ([A2], p. 101). Thus we see that given (x,y)

$$\int K_G(x,nmy)dn = 0,$$

for all $m \in M^1$, unless there is $\gamma \in G(F)$ such that

$$\hat{\tau}_P(H(\gamma x) - H(y) - T_0) = 1.$$

By Lemma 2.3 of [A2], the same assertion is true for each kernel K_χ.

Now let us look at the truncation and the term corresponding to P for the kernel K_G (or one of the kernels K_χ). Fix x in the compact set Ω_0 and y in G^1; suppose that

$$\sum_\delta \hat{\tau}_P(H(\delta y) - T) \int K_G(x, n\delta y)dn \neq 0.$$

Recall the sum is actually finite. Thus there is δ such that

$$\hat{\tau}_P(H(\delta y) - T) = 1 \tag{14}$$

and

$$\int K(x, n\delta y)dn \neq 0.$$

The second relation, in turn, implies that

$$\hat{\tau}_P(H(\gamma x) - H(\delta y) - T_0) = 1.$$

Now ([A1] p. 936) there is a c such that, for all $\omega \in \hat{\Delta}_P \subset \hat{\Delta}_0$:

$$\omega(H(\gamma x)) \leq c(1 + \log \|x\|).$$

Since x is in the compact set Ω_0, we see that

$$\omega(H(\gamma x) \leq c_1$$

for some constant c_1. It follows that

$$\omega(H(\delta y)) < c_1 - \omega(T_0). \tag{15}$$

If T is sufficiently large, the conditions (14) and (15) on $H(\delta y)$ are not compatible. Hence, the term corresponding to $P \neq G$ is 0, if T is sufficiently regular. The first assertion of the proposition follows.

To continue, we recall the explicit expression for K_χ:

$$K_\chi(x, y) = \sum_P n(A_P)^{-1} \int_{\Pi^G(M)} \sum_{\phi \in \mathcal{B}_P(\pi)_\chi} E(x, I_P(\pi, f)\phi).\overline{E(y, \phi)} d\pi. \tag{16}$$

Given X and Y in the enveloping algebra at infinity, we have also a majorization of

$$\sum_\chi \sum_P n(A_P)^{-1} \int_{\Pi^G(M)} \left| \sum_{\phi \in \mathcal{B}_P(\pi)_\chi} R_x(X) R_y(Y) E(x, I_P(\pi, f)\phi)).\overline{E(y, \phi)} \right| d\pi$$

by

$$\|X * f * Y\|_{r_0} \cdot \|x\|^{N_0} \cdot \|y\|^{N_0}, \tag{17}$$

where $\|.\|_{r_0}$ is a certain semi-norm and N_0 an appropriate integer. Thus given Y, we have **a fortiori** a majorization

$$\sum_\chi |R_y(Y) K_\chi(x, y)| \leq \|f * Y\|_{r_0} \cdot \|x\|^{N_0} \cdot \|y\|^{N_0}.$$

This implies that we can truncate the series $K_G = \sum_\chi K_\chi$ term-wise in the second variable:

$$\Lambda_2^T K_G(x,y) = \sum_\chi \Lambda_2^T K_\chi(x,y).$$

We have also, for each integer N, a majorization:

$$\text{(18)} \quad \sum_\chi \sum_P n(A_P)^{-1} \int_{\Pi^G(M)} \left| \sum_{\phi \in \mathcal{B}_P(\pi)_\chi} E(x, I(_P(\pi,f)\phi)).\overline{\Lambda^T E(y,\phi)} \right| d\pi \leq c\|x\|^{N_0}\|y\|^{-N},$$

valid for all $x \in G^1$ and all y in a Siegel set. A fortiori, we obtain an estimate:

$$\sum_\chi |\Lambda_2^T K_\chi(x,y)| \leq c\|x\|^{N_0} \cdot \|y\|^{-N}$$

for all x and all y in a Siegel set. If we take x in a compact set and use the first part of the proposition, we obtain the second part of the proposition. ∎

Exchanging the role of the two variables, we see that for y in a compact set, we can integrate the series $K_G = \sum K_\chi$ with respect to the first variable to obtain:

PROPOSITION 2.2:

$$\int_{U(F)\backslash U(\mathbb{A})} K_G(u,y)du = \sum_\chi \int K_\chi(u,y)du.$$

Moreover, if y is in a compact set, there is $C > 0$ such that

$$\int |K_G(u,y)|du \leq C, \quad \sum_\chi \int |K_\chi(u,y)|du \leq C.$$

We also remark that if we set

$$K_{\mathbf{cusp}}(x,y) = \sum K_\chi(x,y)$$

where the sum is over all cuspidal automorphic representations of G^1, then

$$\int K_{\mathbf{cusp}}(u,x)du$$

may be viewed as the cuspidal component of

$$x \mapsto \int K_G(u,x)du,$$

in the following sense:

PROPOSITION 2.3: *If ϕ is any cusp form, then*

$$\int K_G(u,x)\phi(x)dxdu = \int K_{\mathrm{cusp}}(u,x)\phi(x)dxdu.$$

Proof: Indeed, for fixed u, it follows from the majorization of the Proposition 2.2 that

$$\int K_G(u,x)\phi(x)dx = \sum_{\chi} \int K_\chi(u,x)\phi(x)dx.$$

For a given χ, the integral is 0 unless χ is a cuspidal representation of G^1. Thus

$$\int K_G(u,x)\phi(x)dx = \int K_{\mathrm{cusp}}(u,x)\phi(x)dx.$$

for any u. Thus it suffices to show that the double integrals of K_G and K_{cusp} converge absolutely. This is clear for K_{cusp} because it is bounded. For K_G this will follow from the following assertion:

PROPOSITION 2.4: *There is $c > 0$ and N such that for $x \in G^1$*

$$\int |K_G(u,x)|du \leq c\|x\|^N, \qquad \left|\int K_G(u,x)du\right| \leq c\|x\|^N.$$

Proof: The second assertion follows from the first. We prove the first assertion. There is a function of compact support Φ on $S(\mathbb{A})$ such that

$$\int \Phi(g^* z\xi g)\omega(z)d^\times z = \int f(uzg)du\,\omega(z)d^\times z.$$

Thus

$$\int |K_G(u,x)du| \leq \int \sum_{\sigma\in S(F)} |\Phi(x^* z\bar{z}\sigma x)|d^\times z.$$

Let ϕ be a smooth function of compact support on G whose restriction to S is Φ. As is well known

$$\sum_{\gamma\in G(F)} \int |\phi|(x^{-1}z\bar{z}\gamma y)d^\times z \leq c\|y\|^N.$$

This implies a fortiori our assertion. ∎

Let ψ_F be an additive character of $F_{\mathbb{A}}$, non-trivial but trivial on F. We set as usual

$$\psi_E(z) = \psi_F(z+\bar{z})$$

and define a character θ of the unipotent radical N_0 of P_0 by

$$\theta(n) = \psi_E\left(\sum n_{i,i+1}\right). \tag{19}$$

Our goal is to compute the integral

$$\int_{U(F)\backslash U(F_{\mathbb{A}})\times N_0(E)\backslash N_0(E_{\mathbb{A}})} K_G(u,n)\overline{\theta}(n)du dn.$$

It follows from Proposition 2.2 that we can integrate term-wise to obtain

$$\int K_G(u,n)\overline{\theta} du dn = \sum_{\chi}\int K_{\chi}(u,n)\overline{\theta}(n)du dn.$$

Now for x in a Siegel set and any N, we have a majorization

$$(20)\qquad \sum_{\chi}\sum_{P} n(A_P)^{-1}\int_{\Pi^G(M)}\left|\sum_{\phi\in\mathcal{B}_P(\pi)_{\chi}}\Lambda^T E(x,I(_P(\pi,f)\phi)).\overline{E(y,\phi)}\right| d\pi$$
$$\leq c\|x\|^{-N}.\|y\|^{N_1}$$

for suitable c, N_1. It follows from Proposition 2.3 that if T is sufficiently regular then for any χ

$$(21)\qquad \int K_{\chi}(u,n)\overline{\theta}(n)dndu$$
$$=\sum_P n(A_P)^{-1}\int_{\Pi^G(M)}\sum_{\phi\in\mathcal{B}_P(\pi)_{\chi}}\int\Lambda^T E(u,I(_P(\pi,f)\phi))du\cdot\overline{\int E(n,\phi)\theta(n)dnd\pi}.$$

Now the integral $\int E(n,\phi)\theta(n)dn$ factors through an integral

$$\int_{N(F)\cap M(F)\backslash N(F_{\mathbb{A}})\cap M(F_{\mathbb{A}})}\phi(n_1)\theta(n_1)dn_1.$$

The intersection $N\cap M$ is a maximal unipotent subgroup in M and the restriction of θ to $N\cap M$ is a generic character of that group. The restriction of π to M^1 is in the discrete spectrum of M^1 and this integral is zero unless the representation is cuspidal, because the residual spectrum is degenerate ([MW]), that is, has no Whittaker model; equivalently, the triple $(P, M, \pi|M^1)$ is in the class of χ. We set then

$$(22)\qquad W(\phi,\pi)=\int_{N_0(E)\backslash N_0(E_{\mathbb{A}})} E(n,\phi)\theta(n)dn.$$

Thus we have proved that:

PROPOSITION 2.5:

$$\int K_G(u,n)\overline{\theta}(n)du dn = \sum_{\chi}\int K_{\chi}(u,n)\overline{\theta}(n)du dn.$$

If T is sufficiently regular, for all χ,

$$\int K_\chi(u,n)\overline{\theta}(n)dndu$$

$$= \sum_P n(A_P)^{-1} \int_{(P,M,\pi|M^1)\in\chi} \sum_{\phi\in\mathcal{B}_P(\pi)_\chi} \left(\int \Lambda^T E(u, I(_P(\pi,f)\phi))du\right)\cdot\overline{W(\phi,\pi)}\,d\pi.$$

Our task will be to obtain an explicit expression for the right hand side by letting T tend to infinity. We stress the absolute convergence of the above expression. More precisely, the following sum is finite:

$$\sum_\chi \sum_P n(A_P)^{-1}$$

$$(23) \quad \int_{(P,M,\pi|M^1)\in\chi} \left| \sum_{\phi\in\mathcal{B}_P(\pi)_\chi} \left(\int \Lambda^T E(u, I(_P(\pi,f)\phi))du\right)\cdot\overline{W(\phi,\pi)}\right| d\pi.$$

In what follows, we will use the following notations. We consider a standard parabolic subgroup $P = MN$ in GL(3). The Lie-algebra of A_P is denoted by $\mathfrak{a}_P$. In particular, if $P = G$ then M is the center Z of the group G. We have a canonical splitting:

$$\mathfrak{a}_P = \mathfrak{a}_P^Z \oplus \mathfrak{a}_Z.$$

When no confusion is possible we simply write $\mathfrak{a}$ for $\mathfrak{a}_P^Z$. We consider an automorphic cuspidal representation π of M^1 which we regard as an automorphic representation of $M(E_\mathbb{A})$ trivial on A_P. We assume that the central character of π coincides with ω on the center of G. Suppose that ζ is a complex valued linear form on $\mathfrak{a}$, that is, an element of $\mathfrak{a}_\mathbb{C}^*$. Then we denote by π_ζ the representation of $M(E_\mathbb{A})$ defined by

$$\pi_\zeta(m) = \pi(m_1)e^{\langle\zeta,H(m)\rangle},$$

where $m = m^1a$ with $m^1 \in M^1, a \in A_P$ and $H(m)$ is the logarithm of a. Let $\mathcal{H}_P^0$ be the space of smooth K-finite functions ϕ on $G(E_\mathbb{A})$ such that

$$\phi(nam^1k) = \phi(m^1k)$$

and the function $m^1 \mapsto \phi(m^1k)$ belongs to the space of the automorphic representation π. We recall that because of the multiplicity one theorem, the space is uniquely determined by the class of π. For $\phi \in \mathcal{H}_P^0$ we define the Eisenstein series by analytic continuation of the series:

$$E(g,\phi,\pi_\zeta) = \sum_{\gamma\in P(E)\backslash G(E)} \phi(\gamma g)e^{(\zeta+\rho_P,H(\gamma g))}.$$

If Q is a standard parabolic subgroup associate to P we denote by $\Omega(\mathfrak{a}_P,\mathfrak{a}_Q)$ the Weyl set. If s is in $\Omega(\mathfrak{a}_P,\mathfrak{a}_Q)$ and w_s is a representative, we define the intertwining operator by

$$M(s,\pi_\zeta)\phi(x)e^{(s\zeta+\rho_Q,H(x))}$$
$$= \int_{N_Q(E_\mathbb{A})\cap w_s N_P(E_\mathbb{A})w_s^{-1}\backslash N_Q(E_\mathbb{A})} \phi(w_s^{-1}nx)e^{(s\zeta+\rho_P,H(w_s^{-1}nx))}dn.$$

Often we write $M(s,\zeta)$ for $M(s,\pi_\zeta)$. The space $\mathcal{H}_P^0$ is provided with the scalar product

$$(\phi_1,\phi_2) = \int_{M(E)\backslash M^1\times K} \phi_1(m^1k)\overline{\phi_2}(m_1k)dm^1\,dk.$$

In the previous formula for instance, for a given χ, we sum over all π such that (P,M,π) belongs to χ and then we integrate over $i\mathfrak{a}^*$ for an Euclidean measure $d|z|$. We also sum over an orthonormal basis $\mathcal{B}_P(\pi)$ of $\mathcal{H}_P^0$:

$$\sum_P n(A_P)^{-1}\sum_\pi \int_{i\mathfrak{a}^*}\sum_{\phi\in\mathcal{B}_P(\pi)}\left(\int \Lambda^T E(u,I(_P(\pi,f)\phi))du\right).\overline{W(\phi,\pi)}\,d|\zeta|.$$

It will be useful to keep in mind the following elementary lemma:

LEMMA 2.1: *Let Π be a unitary representation of $G(E_\mathbb{A})$ on a Hilbert space $\mathcal{H}$. Let $\mathcal{H}^0$ be the space of K-finite vectors. Assume that each K-type has finite multiplicity. Let μ and ν be two linear forms on $\mathcal{H}^0$. Consider the expression*

$$\sum_\phi \lambda(\Pi(f)\phi)\overline{\mu(\phi)}$$

where the sum is over an orthonormal basis of $\mathcal{H}^0$ and f is a smooth K-finite function of compact support. Then the sum does not depend on the choice of the orthonormal basis. Moreover, if $f = f_1 * f_2^*$ where f_1 and f_2 are smooth K-finite functions of compact support, and $f_2^*(g) = \overline{f_2(g^{-1})}$ then

$$\sum_\phi \lambda(\Pi(f)\phi)\overline{\mu(\phi)} = \sum_\phi \lambda(\Pi(f_1)\phi)\overline{\mu(\Pi(f_2)\phi)}.$$

The proof is elementary. ∎

3. Estimates for Eisenstein series

We will use estimates for Eisenstein series induced from the minimal parabolic subgroup of GL(3) (or GL(n)) on the imaginary plane (or hyperplane). We recall that in general such estimates are **not** available. The existence of these estimates in the case at hand is a consequence of the fact that we can find zero-free regions for the relevant L-functions.

We fix some notations. Let E be a number field, G_m its multiplicative group, regarded as an algebraic group. Thus G_m^1 is the group of idèles of norm 1. We let A_m be the split component of G_m. This is the group of idèles z such that $z_v = 1$ for v finite, $z_v = a > 0$ for v infinite, where a does not depend on v. Thus A_m is isomorphic to the group $\mathbb{R}^+$. We write in the usual way $G_m(E_{\mathbb{A}}) = E_{\mathbb{A}}^\times$ as the product of G_m^1 and A_m.

We let Π be the group of idèle-class characters of module 1 trivial on A_m. Let U_v be the maximal compact subgroup of $E_v^\times$ and U the product of the groups U_v. We will denote by Π_0 the group of characters $\chi \in \Pi$ which are trivial on U. Let v_i, $0 \le i \le r$, be the infinite places. For $\chi \in \Pi_0$ we have

$$\chi(z_{v_i}) = |z_{v_i}|_{v_i}^{s_i}.$$

The imaginary numbers s_i lie in the hyperplane

$$\sum_i n_i s_i = 0,$$

where n_i is the degree of E_{v_i} on $\mathbb{R}$. They form a lattice, that is, a discrete subgroup of rank $r-1$ in this hyperplane.

In what follows $C_0(\chi)$ will denote a function on Π_0 of the form

$$C_0(\chi) = c\prod_i (1 + s_i\overline{s_i})^{m_i} \tag{24}$$

with $c > 0$, $m_i > 0$. We will introduce furthermore functions on Π of the following form. The functions depend only on the restriction of the character to the subgroup

$$E^\times \cdot E_\infty^\times \prod_v U_v,$$

which has finite index in the idèle group. We choose a set of representative for Π/Π_0. For each such representative χ_0, we choose a function C_{χ_0} of type (24) on Π_0 and then set

$$C(\chi) = C_{\chi_0}(\chi\chi_0^{-1}) \tag{25}$$

if χ has the same restriction as χ_0 to U. Note that there is no uniformity in our choice of C_{χ_0}.

We will also consider integers valued functions $m(\chi)$ with the property that the function depends only on the restriction of χ to U.

PROPOSITION 3.1: *There are functions $C_i(\chi)$, $i = 1,2$ and $m_i(\chi)$, $i = 1,2$ of the above types such that for*

$$s = \sigma + it, \quad |\sigma| \leq C_1(\chi)^{-1}(1+t^2)^{-m_1(\chi)},$$

we have

$$\left|\frac{L(s,\chi)}{L(s+1,\chi)}\right| \leq C_2(\chi)(1+t^2)^{m_2(\chi)}.$$

Moreover, every derivative of the above ratio verifies a similar majorization, with possibly different functions.

SKETCH OF PROOF. The last assertion follows from the first and the integral representation of a derivative via Cauchy formula. We begin the proof of the first assertion. We set

$$L^\infty(s,\chi) = \prod_{v \text{ finite}} L(s,\chi_v).$$

Given a vertical strip of finite width, there are functions $C(\chi)$ and $m(\chi)$ such that for all χ, on the strip

$$|L^\infty(s,\chi)| \leq C(\chi)(1+s\bar{s})^{m(\chi)}.$$

Indeed, if $\Re s$ is sufficiently large, then the product is absolutely convergent hence bounded in a vertical strip. To find estimates for small $\Re s$ we use the functional equation and the Phragmen–Lindelöff principle. Using Cauchy formula, we find similar estimates for the derivatives of $L^\infty(s,\chi)$ on a vertical strip. Next, we use the standard trigonometric identity to prove that

$$\left|L^\infty(s,1)^3 L^\infty(s,\chi)^4 L^\infty(s,\chi^2)\right| \geq 1$$

for $\Re s \geq 0$. This inequality, together with the upper estimates for the derivatives can be used to find (coarse) zero-free regions and lower estimates of the following form: for

$$s = \sigma + it, \quad |\sigma - 1| \leq C_1(\chi)^{-1}(1+t^2)^{-m_1(\chi)} \tag{26}$$

one has

$$|L^\infty(s,\chi)| \geq C_2(\chi)^{-1}(1+t^2)^{-m_2(\chi)}. \tag{27}$$

Of course, the case of a quadratic character must be treated separately; however, there are only finitely many such characters with a given restriction to U and much better estimates are known for characters of finite order. Finally, to prove the proposition one uses Stirling formula to majorize the ratio of the factors at infinity in the proposition. For more details, see for instance [L], p. 313 and p. 334. ∎

To explain our notations, we remark the following: suppose f is a smooth function of compact support on the idèle group which is the product, over all places, of U_v-finite functions. Then, for $C(\chi)$ and $m(\chi)$ of the above type we have:

$$\sum_{\chi\in\Pi} C(\chi)(1+s\bar{s})^{m(\chi)}\left|\int_{E_{\mathbb{A}}^{\times}} f(z)\chi(z)|z|^{s}d^{\times}z\right| < +\infty.$$

Indeed, by hypothesis, the integral is zero unless the restriction of χ to U takes on finitely many possible values. Thus the assertion follows from abelian harmonic analysis.

To continue, we consider an n-tuple of characters $\chi = (\chi_1, \chi_2, \dots, \chi_n)$, each trivial on A_m. We set $\omega = \prod \chi_i$. Let $P_0 = M_0N_0$ be the group of upper triangular matrices, M_0 being the group of diagonal matrices. We let A be the split component of M_0. Then χ defines a character of M_0^1. If π is any permutation of χ then we consider the space $\mathcal{H}_0^0(\pi)$ of K-finite functions ϕ on $G(E_{\mathbb{A}})$ such that

$$\phi(namk) = \phi(k)\pi(m) \tag{28}$$

for $n \in N_0(E_{\mathbb{A}})$, $k \in K$, $a \in A$, and $m \in M_0^1$. If ζ is in $\mathfrak{a}_{\mathbb{C}}^*$ we define the Eisenstein series by analytic continuation of the series

$$E(x,\phi,\pi_\zeta) = \sum_{\delta\in P_0(F)\backslash G(F)} \phi(\delta x)e^{\langle H(\delta x),\rho_0+\zeta\rangle}. \tag{29}$$

Often we drop π from the notation if this does not create confusion. We also denote by $I_0(\pi_\zeta)$ the representation of $G(E_{\mathbb{A}})$ induced by (π_ζ).

Note that the restriction of this representation to K depends only on the restriction of the characters χ_j to U. It space can be viewed as a subspace of $L^2(K)$. We have the intertwining operators $M(s,\pi_\zeta)$ for $s \in \Omega(\mathfrak{a})$. They have no singularity on $i\mathfrak{a}^*$.

We introduce functions of the form

$$C(\chi) = \prod C_{i,j}(\chi_i\cdot\chi_j^{-1}) \prod C_i(\chi_i) \tag{30}$$

and integers $m_i(\chi)$ depending only on the restriction of the characters χ_i to U. We fix a K-type θ, that is, a finite set of classes of irreducible representations of K. We denote by$\|M(s,\pi_\zeta)\|_\theta$ the norm of the intertwining operator in the θ-component of $I_0(\pi_\zeta)$. Note that this space is 0 unless the restriction of the characters χ_j to U belong to a finite set depending on θ.

PROPOSITION 3.2: *There are functions $C_i(\chi)$, $i=1,2$, and $m_i(\chi)$, $i=1,2$, with the following property: let Ω be the open set defined by*

$$|\Re\alpha^\vee(\zeta)| \le C_1(\chi)^{-1}(1+\operatorname{Im}\alpha^\vee(\zeta)^2)^{-m_1(\chi)}, \tag{31}$$

for all $\alpha \in \Delta_0$. Then for $\zeta \in \Omega$ we have

$$\|M(s,\pi_\zeta)\|_\theta \le C_2(\chi)(1+\|\zeta\|^2)^{m_2(\chi)}.$$

Proof: We write M as the product of the normalized intertwining operator and the factor

$$\prod L(\alpha^\vee(\zeta),\chi_i\chi_j^{-1})/L(\alpha^\vee(\zeta)+1,\chi_i\chi_j^{-1}).$$

The product is over all roots $\alpha > 0$ such that $w\alpha < 0$ and $(i,j), i<j$, is the pair of integers corresponding to α. The normalized intertwining operator is easily majorized. The ratio of L-factors is majorized by the previous proposition. Our assertion follows. ∎

As before, we obtain similar estimates for any derivative of the intertwining operator.

PROPOSITION 3.3: *Fix a K-type θ. Then there exists $C_i(\chi)$ and $m_i(\chi)$ with the following property: consider the open set Ω defined by* (31)*; then for ζ in Ω and ϕ in the θ-component of the induced representation*

$$\int \left|\Lambda^T E(x,\phi,\pi_\zeta)\right|^2 dx \tag{32}$$

$$\le \|\phi\|^2(1+\log(\|T\|))^{m_2(\chi)}e^{m_3(\chi)\|T\|}C_2(\chi)(1+\|\zeta\|^2)^{m_4(\chi)}.$$

Proof: This follows from the scalar product formula ([A3]) and the estimates of the intertwining operator and its derivatives. ∎

PROPOSITION 3.4: *Let f be a smooth function of compact support on $G(E_{\mathbb{A}})$ which is K-finite. Then there are functions $C_i(\chi)$ and $m_i(\chi)$ such that on the open set Ω defined by* (31) *we have for all $x \in G^1$*

$$|E(x,I_0(\pi_\zeta,f)\phi,\pi_\zeta)| \le \|\phi\|C_2(\chi)(1+\|\zeta\|^2)^{m_2(\chi)}\|x\|^{m_3(\chi)}. \tag{33}$$

Proof: From [A1], page 936, we recall there is a constant c such that for any $\omega \in \hat{\Delta}_0$ and $\gamma \in G(E)$ we have

$$\omega(H(\gamma x)) \leq c(1+\log\|x\|).$$

In particular, suppose Ω_0 is a compact set of G^1. Then if T is so chosen that

$$\omega(T) \geq c(1+\log\|x\|)$$

for all ω and all $x \in \Omega_0$, we have

$$\Lambda^T\phi(x) = \phi(x)$$

for all functions ϕ and all $x \in \Omega_0$. Choose T_1 sufficiently regular such that

$$\omega(T_1) \geq c$$

for all $\omega \in \hat{\Delta}_0$ and let c_2 be its norm. Let us set

$$T = T_1 \sup_{\Omega_0}(1+\log\|x\|).$$

Then T satisfies the above condition. Moreover

$$\|T\| = c_2 \sup_{\Omega_0}(1+\log\|x\|).$$

Let Ω_2 be the support of f and Ω_1 be a compact subset of G^1. Let us apply the previous construction to the compact set $\Omega_0 = \Omega_1\Omega_2$. With the above T we have now

$$\|T\| \leq c_3 \sup_{\Omega_1}(1+\log\|x\|)$$

and

$$\int \Lambda^T E(xy,\phi,\pi_\zeta)f(y)dy = \int E(xy,\phi,\pi_\zeta)f(y)dy = E(x, I_0(\pi_\zeta,f)\phi,\pi_\zeta) \tag{34}$$

for all $x \in \Omega_1$. Set

$$K_f(x,y) = \int \sum_{\gamma\in G(E)} f(x^{-1}z\gamma y)\,\omega(z)d^\times z.$$

Then the first integral in (34) is also

$$\int \Lambda^T E(y,\phi,\pi_\zeta)K_f(x,y)dy. \tag{35}$$

Since

$$|K_f(x,y)| \le c\|x\|^N$$

we find this is bounded by a constant, times a power of $\|x\|$, times the L^2 norm of the truncated Eisenstein series. Taking in account the previous proposition and the upper estimate for the norm of T, we see that we obtain a majorization of (34) by

$$C(\chi)(1+\|\zeta\|^2)^{m_1(\chi)}\|\phi\|(1+\sup\log\|x\|)^{m_2(\chi)}\sup\|x\|^{m_3(\chi)},$$

where the supremun is over the compact set $\Omega_1 \ni x$. Let Ω_3 be a compact neighborhood of the origin in G^1. Fix $x_0 \in G^1$ and consider the compact set $\Omega_1 = x_0\Omega_3$. Then the supremun of $\|x\|$ over Ω_1 is bounded by a constant times $\|x_0\|$. Applying the above majorization for x_0 and Ω_1 we obtain our result. ∎

Finally, we obtain the required majorization:

PROPOSITION 3.5: *Let f be a smooth function of compact support which is also K-finite. Let D be a differential operator with constant coefficients on $\mathfrak{a}$. Then there are functions $C(\chi)$ and $m_i(\chi)$ such that if π is a permutation of χ, then, for all ϕ, and all $\zeta \in i\mathfrak{a}^*$:*

$$|DE(x, I_0(\pi_\zeta, f)\phi, \pi_\zeta)| \le C(\chi)(1+\|\zeta\|^2)^{m_1(\chi)}\|x\|^{m_2(\chi)}\|\phi\|. \tag{36}$$

4. Estimates for an infinite series

In order to compute formally the integral of a truncated Eisenstein series over the unitary group, we will need to replace the terms in Arthur's second formula for the truncation by their absolute value and show that the result is integrable over the unitary group. In this section, we estimate an infinite series. The result will be used in the next section to majorize the series of absolute values.

We let F be a number field; we fix a constant $C_0 > 0$ and we consider the following series:

$$\sum_{\xi_2,\xi_3,\dots,\xi_n \in F} \|(a_1, a_2\xi_2, \dots, a_n\xi_n)\|^{-s}, \tag{37}$$

where $a_1, a_2, ..., a_n$ are in the split component of $G_m = \mathrm{GL}(1)$ and

$$|a_1| \ge C|a_j|, \quad \text{for all } j \ge 2;$$

the sum is restricted by the condition

$$\|(a_1, a_2\xi_2, \ldots, a_n\xi_n)\| \geq R. \tag{38}$$

We recall that the split component of GL(1) is the set of idèles z such that $z_v = 1$ for v finite and $z_v = a$, for v infinite, where $a > 0$ does not depend on v. For a primitive vector x we denote by $\|x\|$ the product

$$\prod_v \|x_v\|,$$

where

$$\|x_v\| = \sup_i |x_{v,i}|_v$$

if v is finite,

$$\|x_v\| = \sqrt{\sum_i x_{v,i}^2}$$

if v is real and

$$\|x_v\| = \sum_i x_{v,i}\overline{x}_{v,i}$$

if v is complex.

LEMMA 4.1: *The previous sum converges for $s > n$ and is then bounded by*

$$C(s)|a_1a_2\cdots a_n|^{-1}$$

where the constant $C(s) > 0$ is a locally bounded function of s, which depends on R and C_0. For $F = \mathbb{Q}$ we can take $C(s) = C_1(s)R^{n-s}$ where $C_1(s)$ does not depend on R.

Proof: It suffices to prove the result when $F = \mathbb{Q}$. Indeed, if F has degree m over $\mathbb{Q}$ we choose a basis τ_j, $1 \leq j \leq m$, of F over $\mathbb{Q}$. For convenience, we may assume $\tau_1 = 1$. Then every ξ_i has coordinate (ξ_i^j). Thus the vector

$$x = (a_1, a_2\xi_2, \ldots, a_n\xi_n)$$

in $F_{\mathbb{A}}^n$ determines a vector

$$y = (a_1, 0, 0, \ldots, 0, a_2\xi_2^1, a_2\xi_2^2, \ldots, a_2\xi_2^m, \ldots, a_n\xi_n^1, a_n\xi_n^2, \ldots, a_n\xi_n^n)$$

in $\mathbb{A}^{mn}$. Furthermore

$$c_1\|y\| \leq \|x\| \leq c_2\|y\|^m,$$

for suitable constants c_1, c_2. Ignoring the constants, we see that the original sum is bounded by

$$\sum \|y\|^{-s}$$

where the sum is restricted by the condition that

$$\|y\| \geq R^{1/m}.$$

In turn, this is majorized by the sum of all terms of the form

$$\|(a_1, a_1\zeta_1, a_1\zeta_2, \ldots, a_1\zeta_m, a_2\xi_2^1, a_2\xi_2^2, \ldots, a_2\xi_2^m, \ldots, a_n\xi_n^1, a_n\xi_n^2, \ldots, a_n\xi_n^m)\|^{-s},$$

such that the norm is $\geq R^{1/m}$. Applying the result over $\mathbb{Q}$ we find our result.

Thus we may assume $F = \mathbb{Q}$. By homogeneity, we may assume $a_1 = 1$. Let Ω be a relatively compact neighborhood of 0 of $\mathbb{A}$ such that $\Omega \cap \mathbb{Q} = 0$. Since $|a_i| \leq C^{-1}$ the sets $a_i\Omega$ are relatively compact. Let g be the $n \times n$ matrix with unit diagonal, first row

$$(1, a_2\omega_2, \ldots, a_n\omega_n),$$

the other entries being 0. Then g remains in a compact subset. Now

$$(1, a_2\xi_2, \ldots, a_n\xi_n)g = (1, a_2(\xi_2 + \omega_2), \ldots, a_n(\xi_n + \omega_n)).$$

It follows that the ratio of the norms of the vectors

$$(1, a_2(\xi_2 + \omega_2), \ldots, a_n(\xi_n + \omega_n))$$

and

$$(1, a_2\xi_2, \ldots, a_n\xi_n)$$

remains in a compact set of $\mathbb{R}^\times$. Our series is thus bounded by a constant times the integral

$$\int_{\Omega^{n-1}} \|(1, a_2(\xi_2 + \omega_2), \ldots, a_n(\xi_n + \omega_n))\|^{-s} \otimes d\omega_i,$$

where the integral is over the set of vectors such that

$$\|(1, a_2(\xi_2 + \omega_2), \ldots, a_n(\xi_n + \omega_n))\| \geq c_1 R.$$

The constant c_1 depends only on Ω. In turn, this integral is less than the corresponding integral over $\mathbb{A}^{n-1}$. Thus we are reduced to the problem of estimating the integral

(39) $$\int \|(1, a_2x_2, \ldots, a_nx_n)\|^{-s} \otimes dx_i,$$

taken over the set of vectors such that

$$\|(1, a_2x_2, \ldots, a_nx_n)\| \geq R.$$

We have to see that the integral is bounded by

$$C(s)R^{n-s}|a_2a_3\cdots a_n|^{-1}.$$

After a change of variables, we see that we may assume $a_i = 1$.

We see that our integral takes the form

$$\int_{\|(1,x)\|\geq R} \|(1,x)\|^{-s}dx. \tag{40}$$

We write $x = x_\infty x^\infty$ where x^∞ the product of the x_v for v finite. The integral can be written as

$$\int \|(1,x^\infty)\|^{-s}dx^\infty \int \|(1,x_\infty)\|^{-s}dx_\infty,$$

where the inner integral is for

$$\|(1,x_\infty)\| \geq R\|(1,x^\infty)\|^{-1}.$$

The domain of integration is contained in the union of the two following sets:

$$\|(1,x_\infty)\| \geq R\|(1,x^\infty)\|^{-1}, \qquad \|(1,x^\infty)\| \leq R/2, \tag{41}$$

$$\|(1,x^\infty)\| > R/2. \tag{42}$$

Thus we majorize our integral by the sum of the integrals over these two sets. Let us consider the second integral. The integral is now the product of an integral at infinity and an integral over the finite adèles. The integral at infinity converges, so we are left with the task of estimating an integral of the form

$$\int_{\|(1,x^\infty)\|>R} \|(1,x^\infty)\|^{-s}dx^\infty. \tag{43}$$

It can be computed as

$$\prod_p \left(1 + \sum_{j_p\geq 1} p^{((n-1)-s)j_p}(1-p^{-n+1})\right),$$

where in the expansion of the product we keep only the terms such that

$$\prod p^{j_p} > R.$$

We obtain a larger expression by ignoring the factor $1 - p^{-n+1}$. Thus we have majorized the integral over (42) by the product of a bounded function of s and

$$\sum_{m \in \mathbb{N},\ m > R} m^{n-1-s}.$$

In turn this is less that $C(s)R^{n-s}$, where $C(s)$ does not depend on R and is locally bounded.

We now majorize similarly the integral over (41). We first integrate the component at infinity. Using polar coordinates we obtain

$$\int r^{n-2}(1+r^2)^{-s/2} dr$$

with

$$r \geq R\|(1, x^\infty)\|^{-1}\sqrt{1 - \|(1, x^\infty)\|^2 R^{-2}}.$$

Recall we are taking

$$R\|(1, x^\infty)\|^{-1} \geq 2.$$

Thus the square root is bounded below by a positive constant c and we obtain a larger integral by integrating over

$$r \geq Rc\|(1, x^\infty)\|^{-1}.$$

Furthermore, the integrand is bounded by r^{n-2-s}. Thus the integral at infinity is bounded by a locally bounded function $C(s)$ times

$$R^{n-1-s}\|(1, x^\infty)\|^{s+1-n}.$$

Thus we are left with the task of estimating the integral

$$\int \|(1, x^\infty)\|^{1-n} dx^\infty$$

taken over $\|(1, x^\infty)\| \leq R/2$. As before this is bounded by

$$\sum_{m \leq R/2} m^{1-n}$$

which, in turn, is bounded by a constant times R and we are done. ∎

We will also consider sums of the form

$$\sum_{\xi_2, \xi_3, \ldots, \xi_n \in F} \|(a_1, a_2\xi_2, \ldots, a_n\xi_n)\|^s, \tag{44}$$

where $a_1, a_2, \ldots, a_n$ are in the split component of GL(1) and $|a_1| \geq C_0|a_j|$ and the sum is restricted by the condition

$$\|(a_1, a_2\xi_2, \ldots, a_n\xi_n)\| \leq R. \tag{45}$$

LEMMA 4.2: *For $s \geq 0$, the sum is bounded by $C(s)|a_1a_2\cdots a_n|^{-1}$, where $C(s)$ is a locally bounded function.*

Proof: We apply the same reduction. We find we may assume $F = \mathbb{Q}$. Then we need to majorize the integral

$$\int_{\|(1,x)\|\leq R} \|(1,x)\|^s dx$$

by $C(s)R^{n+s}$. The integrand being bounded by R^s, it suffices to show that the volume of the compact set $\|(1,x)\| \leq R$ is bounded by a constant times R^n. Thus we have to majorize the integral

$$\int dx^\infty \int dx_\infty$$

taken over the set

$$\|(1, x_\infty)\| \leq R\|(1, x^\infty)\|^{-1}\ ,\ \|(1, x^\infty)\| \leq R.$$

We can use polar coordinates to evaluate the integral at infinity. We obtain a larger domain of integration by integrating over

$$r \leq R\|(1, x^\infty)\|^{-1}.$$

We obtain a majorization of the integral by

$$R^{n-1}\|(1, x^\infty)\|^{1-n}.$$

Finally, we have to estimate

$$R^{n-1}\int_{\|(1,x^\infty)\|\leq R} \|(1, x^\infty)\|^{-n+1}dx.$$

As before it is bounded by a constant times R^n and we are done. ∎

5. Majorization of the truncated Eisenstein series

We now consider the group $G = \mathrm{GL}(3)$. We write $P_0 = M_0 N_0$ for the parabolic subgroup of type $(1,1,1)$. Here M_0 is the group of diagonal matrices, $\Delta_0 = \{\alpha_1, \alpha_2\}$ is the set of simple roots and $\hat{\Delta}_0 = \{\omega_1, \omega_2\}$ the set of fundamental weights. In particular, if we denote by e_i the canonical basis of the space of row vectors, we have

$$e^{(\omega_2, H(g))} = \|e_3 g\|^{-1}, \quad e^{(\omega_1, H(g))} = \|e_2 g \wedge e_3 g\|^{-1}. \tag{46}$$

We let A_0 be the split component of M_0. The simple reflections are noted s_1 and s_2. We set $s_0 = s_1 s_2 s_1$. Thus the Weyl group is given by

$$\Omega(\mathfrak{a}_0) = \{e, s_1, s_2, s_1 s_2, s_2 s_1, s_0\}.$$

Let $f(g,\zeta)$ be the function on $G(F_{\mathbb{A}}) \times \mathfrak{a}^*$ defined by

$$f(g,\zeta) = \exp\left(H(g), \zeta + \rho_0\right).$$

We fix integers $m_1 > 0$ and $m_2 > 0$ sufficiently large and then for all T sufficiently regular we define a function $f_T(g,\zeta)$ in the following way: if, for $i = 1, 2$, we have

$$(H(g), \omega_i) \leq (T, \omega_i) \text{ and } (\zeta, \alpha_i^\vee) > m_1 (\rho_0, \alpha_i^\vee), \tag{47}$$

or

$$(H(g), \omega_i) > (T, \omega_i) \text{ and } (\zeta, \alpha_i^\vee) < -m_2 (\rho_0, \alpha_i^\vee), \tag{48}$$

then $f_T(g,\zeta) = f(g,\zeta)$; otherwise $f_T(g,\zeta) = 0$.

LEMMA 5.1: *On a Siegel set the series*

$$\sum_{\xi \in P_0(F) \backslash G(F)} f_T(\xi g, s)$$

is bounded by a constant multiple of

$$e^{(H(g),\omega_1)} + e^{(H(g),\omega_2)}.$$

Proof: We first remark that if Ω is a compact set there is c and T_1 such that $T - T_1$ is in a compact set and

$$f_T(g\omega, s) \leq c f_{T_1}(g, s)$$

for all g and all $\omega \in \Omega$. Thus, it suffices to find a majorization of the series on matrices in A_0 of the form

$$a = \mathrm{diag}(a_1, a_2, a_3)$$

with

$$|a_1| \geq c_1|a_2| \geq c_2|a_3| \ , \ |a_1a_2a_3| = 1,$$

for some constants c_1, c_2. We set $T_i = \omega_i(T)$ and $\zeta_i = \omega_i(\zeta)$.

We use the Bruhat decomposition to write down the contribution of each coset sN_0 to the series. The contribution of e is trivially bounded by a constant. For simplicity, we write the contribution of s_1, s_2s_1, s_0. The other terms are treated similarly. For s_0 we find

$$\sum \|(a_1, a_2\xi_2, a_3\xi_3)\|^{-\zeta_2-1}\|(a_1a_2, a_1a_3\xi_1, a_2a_3\xi_3')\|^{-\zeta_1-1},$$

where $\xi_3'+\xi_3 = \xi_1\xi_2$. The sum is restricted by the following condition: if $\zeta_2 > m_1$ then we demand that the norm of the first vector be $\geq e^{-T_2}$; if $\zeta_2 < -m_2$ we demand that the norm of the first vector be $< e^{-T_2}$; if $-m_2 \leq \zeta_2 \leq m_1$ the sum is empty; similarly for the second vector. We can majorize this sum by the same sum but with ξ_3' independent of the other variables. This new sum is now a product

$$\sum_{\xi_2,\xi_3} \|(a_1, a_2\xi_2, a_3\xi_3)\|^{-\zeta_2-1} \times \sum_{\xi_1,\xi_3'} \|(a_1a_2, a_1a_3\xi_1, a_2a_3\xi_3')\|^{-\zeta_1-1}.$$

Each sum is restricted by the same condition as before. If we take $m_1 \geq 2$ it follows from the basic lemma that each sum is bounded above by a constant (for ζ in a compact set).

We pass to the contribution of the element s_2s_1. It can be written as

$$\sum_{\xi_2,\xi_3} \|(a_1, a_2\xi_2, a_3\xi_3)\|^{-\zeta_2-1}\|(a_1a_3, a_2a_3\xi_2)\|^{-\zeta_1-1}.$$

The sum is restricted by conditions similar to the previous one. Again this sum is majorized by the product of two independent sums:

$$\sum_{\xi_2,\xi_1} \|(a_1, a_2\xi_2, a_3\xi_1)\|^{-\zeta_2-1} \times \sum_{\xi_3} \|(a_1a_3, a_2a_3\xi_3)\|^{-\zeta_1-1},$$

where each sum is restricted by the same kind of conditions. The first sum is simply bounded above by a constant (for ζ in a compact set). The second sum is bounded by a constant multiple of

$$|a_3|^{-1} = |a_1a_2| = \exp\left(H(a), \omega_2\right).$$

Finally the contribution of s_1 is written as

$$\sum_{\xi} \|a_3\|^{-\zeta_2-1} \|(a_1a_3, a_2a_3\xi)\|^{-\zeta_1-1}.$$

The sum is restricted by the same condition as before. Thus the first factor is bounded above and the second factor is again bounded by

$$|a_3|^{-1} = |a_1a_2| = \exp\left(H(a), \omega_2\right).$$

This concludes the proof. ∎

Finally, we consider a parabolic subgroup P of type $(2,1)$ or $(1,2)$. Thus $\Delta_P = \{\alpha\}$, $\hat{\Delta}_P = \{\omega\}$. For $\zeta \in \mathfrak{a}_P^*$ we define

$$f(g,\zeta) = \exp\left(H(g), \zeta + \rho_P\right). \tag{49}$$

If T is suitably regular, we define a new function f_T by demanding that it be zero, unless the following condition is satisfied, in which case it is equal to $f(g,\lambda)$:

$$(\zeta, H(g)) \leq (\zeta, T) \quad \text{and} \quad (\zeta, \alpha^\vee) > m_1(\rho_P, \alpha^\vee) \tag{50}$$

or

$$(\zeta, H(g)) < (\zeta, T) \quad \text{and} \quad (\zeta, \alpha^\vee) < -m_2(\rho_P, \alpha^\vee). \tag{51}$$

LEMMA 5.2: *For g in a Siegel set the series*

$$\sum_{P(F)\backslash G(F)} f_T(\gamma g, \zeta)$$

is bounded by a constant multiple of

$$e^{(H(g),\omega_1)} + e^{(H(g),\omega_2))}.$$

The proof is similar (but simpler). ∎

We consider now a standard parabolic subgroup $P_1 = M_1N_1$ in $\mathrm{GL}(3)$ and an Eisenstein series induced from cusp forms belonging to a cuspidal representation

π_1 of M_1. It is noted $E(x,\phi,\zeta)$. We consider the truncation of this Eisenstein series and Arthur's second formula for the truncation:

$$\Lambda^T E(x,\phi,\zeta) \tag{52}$$

$$= \sum_{P_2} \sum_{\delta\in P_2(E)\backslash G(E)} \sum_{s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_2)} \epsilon_2(s\zeta_R)\phi_2(s\zeta_R, H(\delta x)-T))$$

$$\cdot \exp\left(s\zeta+\rho_2\right)(H(\delta x))\left(M(s,\zeta)\phi(\delta x)\right).$$

To insure convergence, the real part ζ_R of ζ is assumed to satisfy

$$\zeta_R(\alpha^\vee) > \rho_1(\alpha^\vee),$$

for all $\alpha \in \Delta_1$. We recall that $\epsilon_2 = \pm 1$, that $H \mapsto \phi_2(s\zeta_R, H)$ is the characteristic function of a certain subset of $\mathfrak{a}_2$ and $M(s,\zeta)$ is the intertwining operator. The relevant information about ϵ_2 and ϕ_2 is recalled below. We now consider the series obtained by replacing each term by its absolute value:

$$\sum_{P_2} \sum_{\delta\in P_2(E)\backslash G(E)} \sum_{s\in\Omega(\mathfrak{a}_1,\mathfrak{a}_2)} \phi_2(s\zeta_R, H(\delta x)-T) \tag{53}$$

$$\cdot \left|\exp\left(s\zeta+\rho_2\right)(H_0(\delta x))\left(M(s,\zeta)\phi(\delta x)\right)\right|.$$

We will call this series of positive terms the dominating series of the truncated Eisenstein series. We will choose an integer m sufficiently large. Then:

PROPOSITION 5.1: *Assume that the real part of ζ satisfies*

$$\zeta_R(\alpha^\vee) > m\rho_1(\alpha^\vee),$$

for all $\alpha \in \Delta_1$. Then for ζ in a compact set and g in a Siegel set, the dominating series of the truncated Eisenstein series is bounded by a constant multiple of

$$e^{(H(g),\omega_1)} + e^{(H(g),\omega_2)}.$$

Proof: Assume $P_1 = P_0$. We write $\zeta = \zeta_1\omega_1 + \zeta_2\omega_2$. The following table gives the value of $\epsilon_2(s\zeta_R)$, $s\zeta$ and the inequalities defining the set

$$\{x\colon \phi_2(s\zeta_R, H_0(x)-T) = 1\}.$$

s	ϵ_2	$s\zeta$	$(\omega_1, H(x))$	$(\omega_2, H(x))$
e	+	$\zeta_1\omega_1 + \zeta_2\omega_2$	$\leq T_1$	$\leq T_2$
s_1	-	$-\zeta_1\omega_1 + (\zeta_1 + \zeta_2)\omega_2$	$> T_1$	$\leq T_2$
s_2	-	$(\zeta_1 + \zeta_2)\omega_1 - \zeta_2\omega_2$	$\leq T_1$	$> T_2$
$s_2 s_1$	-	$\zeta_2\omega_1 - (\zeta_1 + \zeta_2)\omega_2$	$\leq T_1$	$> T_2$
$s_1 s_2$	-	$-(\zeta_1 + \zeta_2)\omega_1 + \zeta_1\omega_2$	$> T_1$	$\leq T_2$
s_0	+	$-\zeta_2\omega_1 - \zeta_1\omega_2$	$> T_1$	$> T_2$

If we take $m = 2$ we see that the dominating series is itself bounded by a sum of series

$$\sum_{\gamma \in P_0(F)\backslash G(F)} f_T(g, \zeta_R)$$

for $m_1 = m$ and a suitable $m_2 > 0$. The conclusion follows from the previous lemma. The case of a maximal parabolic subgroup is similar. ∎

Consider now the quasi-split unitary group U which fixes the Hermitian matrix

$$\sigma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{54}$$

Our task will be to obtain a formula for the integral of a truncated Eisenstein series over $U(F)\backslash U(F_{\mathbb{A}})$. It is clear that the integral converges since the truncated Eisenstein series is bounded and U is semi-simple. We would like to replace the truncated Eisenstein series by Arthur's second formula and compute formally. We assume that the condition of the previous proposition is satisfied:

$$\zeta_R(\alpha^\vee) > m\rho_1(\alpha^\vee)$$

for all $\alpha \in \Delta_1$. To show that the formal computation is justified, we may replace the truncated Eisenstein series by its majorizing series and show the majorizing series is integrable over $U(F)\backslash U(F_{\mathbb{A}})$. This will follow from the previous estimate and the following lemma:

LEMMA 5.3: *Let $\mathfrak{S}$ be a Siegel set for G relative to the parabolic subgroup P_0. Then $\mathfrak{S}_U = \mathfrak{S} \cap \mathfrak{U}$ is a Siegel set for the parabolic subgroup $P_0 \cap U$ of U. The function*

$$g \mapsto \exp\left(H(g), \omega_i\right)$$

is integrable over the Siegel set $\mathfrak{S}_U$ of U.

Proof: The intersection of P_0 with U is a maximal parabolic subgroup of U defined over F with Levi-decomposition:

$$P_0 \cap U = (M_0 \cap U).(N_0 \cap U).$$

The first assertion follows at once. We denote by H_U and ρ_U the analogue of H and ρ for the group U. We have, for $i = 1, 2$ and $u \in U$,

$$e^{(H(u),\omega_i)} = e^{(H_U(u),\rho_U)}.$$

The function on the right is clearly integrable over the Siegel set of U. ∎

6. Integral of truncated Eisenstein series: I

Let $\xi \in S(F) \subset \mathrm{GL}(n, E)$ and $U = U_\xi$ be the corresponding unitary group. Since a truncated Eisenstein series is rapidly decreasing, it is integrable over $U(F)\backslash U(F_{\mathbb{A}})$. Recall our task is to compute this integral (for $n = 3$). In this section we study this question from a formal point of view. It is best to do this in the context of $\mathrm{GL}(n)$. By the results of the previous sections, the computation will be justified for $n = 3$. Thus we let ξ be an arbitrary invertible Hermitian matrix. We let S_ξ be the variety of matrices equivalent to ξ. We consider the polarization map

$$g \mapsto g \cdot \xi \cdot g^* \tag{55}$$

from G onto S_ξ and the action of G:

$$(g, s) \mapsto gsg^*.$$

According to a result of Springer ([t.S]) the orbits of $N_0(E)$ on S_ξ have a set of representatives whose elements are product of a permutation matrix of order 1 or 2 and a diagonal matrix. Thus if P is another standard parabolic subgroup, each orbit of P in S admits a representative of this form. Of course, two such matrices may be in the same orbit of $P(E)$. It follows that

$$P(E)\backslash G(E) = \bigcup \eta_w T'_w(F)\backslash U(F), \tag{56}$$

where η_w maps to w under the polarization map and

$$T'_w = \eta_w^{-1} P(E) \eta_w \cap U(F). \tag{57}$$

The union is over a set of representatives w of the above type for the orbits of $P(E)$. We also introduce

$$T_w = P(E) \cap \eta_w U(F) \eta_w^{-1}. \tag{58}$$

Thus T_w is the stabilizer of w for the action of P on S.

Let ϕ be a continuous function on $N(E_{\mathbb{A}})P(E)\backslash G(E_{\mathbb{A}})$. Set

$$\Phi(g) = \sum_{\gamma \in P(E)\backslash G(E)} \phi(\gamma g). \tag{59}$$

For the moment we assume that the series

$$\sum_{\gamma \in P(E)\backslash G(E)} |\phi(\gamma g)|$$

converges absolutely and is integrable over $U(F)\backslash U(F_{\mathbb{A}})$. In particular, the following formal computations are justified. We compute

$$\int \Phi(u) du. \tag{60}$$

It is equal to

$$\sum_w \int_{T'_w(F_{\mathbb{A}})\backslash U(F_{\mathbb{A}})} du \int_{T_w(F)\backslash T_w(F_{\mathbb{A}})} \phi(t\eta_w u) dt. \tag{61}$$

The group T_w need not be uni-modular; thus in the above integral, dt is the left invariant measure on $T(F_{\mathbb{A}})$; after integrating over t the resulting function of u transforms under the module of the group T'_w and du denotes a linear form, invariant on the right, on the space of functions transforming on the left under this module. In this section, our task is to find which elements w contribute effectively to this sum. The result (Proposition 6.1) is similar to a result of Arthur–Langlands giving the scalar product of a truncated Eisenstein series induced from cusp forms with itself: if the Eisenstein series is constructed from a parabolic subgroup P then, a priori, all associate P_i of P and all double cosets $P_i\backslash G/P_j$ contribute to the scalar product formula; however, it is found that all only the double cosets with a representative in $\Omega(\mathfrak{a}_i, \mathfrak{a}_j)$ contribute.

Fix a w and set

$$\theta(g) = {}^{\theta}g = w\,{}^t\overline{g}^{-1} w^{-1}.$$

Thus θ is an automorphism of order 2 of G (defined over F when we regard G as an F-group by restriction of scalars) and the group of fixed points of θ is U_w.

In particular, ${}^{\theta}P$ is a semi-standard parabolic subgroup of G equal to wPw^{-1} and g is in T_w if and only if g is in $P \cap^{\theta} P$ and, in addition, $g = \theta(g)$. We let $P = MN$ be the Levi decomposition of P with $M \supseteq M_0$. Then

$$ {}^{\theta}P = {}^{\theta}M\, {}^{\theta}N = (wMw^{-1}).(wNw^{-1}) $$

is the Levi decomposition of ${}^{\theta}P$ with ${}^{\theta}M \supseteq M_0$. We recall standard results on the intersection of two parabolic subgroups:

LEMMA 6.1: *The group*

$$ V = (M \cap {}^{\theta}N) \cdot (N \cap {}^{\theta}M) \cdot (N \cap {}^{\theta}N) $$

is a unipotent subgroup. The subgroups

$$ N \cap {}^{\theta}N, \quad (N \cap {}^{\theta}M) \cdot (N \cap {}^{\theta}N), \quad (M \cap {}^{\theta}N) \cdot (N \cap {}^{\theta}N) $$

are normal subgroups of V. The commutator of an element of $M \cap {}^{\theta}N$ and an element of ${}^{\theta}M \cap N$ is in $N \cap {}^{\theta}N$. Finally $V = \theta(V)$ and we have a semi-direct product:

$$ P \cap {}^{\theta}P = (M \cap {}^{\theta}M)V. $$

In particular $M \cap {}^{\theta}P$ is a parabolic subgroup of M with unipotent radical $M \cap {}^{\theta}N$.

This allows us to have a simple description of T_w. We let V_w be the intersection of T_w and V and similarly, we let M_w be the intersection of M and T_w. Thus M_w is in fact the group of points of M fixed by θ and V_w is the group of points of V fixed by θ.

LEMMA 6.2: *We have a semi-direct product:*

$$ T_w = M_w V_w, $$

where M_w is reductive and V_w unipotent. An element $v \in V$ is in V_w if and only if it is of the form

$$ v = v_1 \theta(v_1) u_1 u_2, \tag{62} $$

where v_1 is an element of $M \cap^{\theta} N$, and $u_1 \in N \cap {}^{\theta}N$ is a solution of the equation

$$ u_1 \theta(u_1)^{-1} = \theta(v_1)^{-1} v_1^{-1} \theta(v_1) v_1, \tag{63} $$

and u_2 is an element of $N \cap {}^{\theta}N$ fixed by θ. Moreover, given $v_1 \in M \cap {}^{\theta}N$ and $u_2 \in N \cap {}^{\theta}N$ fixed by θ the equation (63) has a solution in $N \cap {}^{\theta}N$ and the element v defined by (62) is in V_w.

Proof: Let g be in $P \cap {}^\theta P$. Then

$$g = mv, \quad m \in M \cap {}^\theta M, \quad v \in V.$$

Thus $\theta(g) = \theta(m)\theta(v)$. In particular $g = \theta(g)$ if and only if $m = \theta(m)$ and $v = \theta(v)$. This establishes the first assertion.

Let us write $v \in V$ in the form

$$v = v_1 v_2 u$$

with $v_1 \in M \cap {}^\theta N$, $v_2 \in {}^\theta M \cap N$, $u \in N \cap {}^\theta N$. Then

$$\theta(v) = \theta(v_1)\theta(v_2)\theta(u) = \theta(v_2)\theta(v_2)^{-1}\theta(v_1)\theta(v_2)\theta(u).$$

Now $\theta(v_2)$ is in $M \cap {}^\theta N$ and $\theta(v_2)^{-1}\theta(v_1)\theta(v_2)$ in $(M \cap {}^\theta N) \cdot (N \cap {}^\theta N)$. Now suppose that $v = \theta(v)$. Then $\theta(v_2) = v_1$. Hence we see that

$$v = v_1 \theta(v_1) u$$

and

$$u\theta(u)^{-1} = \theta(v_1)^{-1} v_1^{-1} \theta(v_1) v_1.$$

Conversely, let $v_1 \in M \cap {}^\theta N$ be given. Then

$$s = \theta(v_1)^{-1} v_1^{-1} \theta(v_1) v_1$$

is in $N \cap {}^\theta N$ and verifies $\theta(s) = s^{-1}$. Since θ defines an automorphism of order 2 of the unipotent group $N \cap {}^\theta N$, there is an element u of $N \cap {}^\theta N$ such that

$$s = u\theta(u)^{-1}.$$

Then

$$v = v_1 \theta(v_1) u$$

is fixed by θ. This concludes the proof of the lemma. ∎

LEMMA 6.3: *Let N_w be the group of points on $N \cap {}^\theta N$ fixed by θ. Then*

$$\int_{M \cap {}^\theta N(F_{\mathbb{A}})} dv_1 \int_{N_w(F_{\mathbb{A}})} \phi(u_2 u_1 \theta(v_1) v_1) du_2$$

is an invariant measure on $V_w(F_{\mathbb{A}})$. In this formula, du_2 is the standard Haar measure on $N_w(F_{\mathbb{A}})$, dv_1 the standard measure on $M \cap^\theta N(F_{\mathbb{A}})$, u_1 any element of $N \cap {}^\theta N$ such that the product $u_1\theta(v_1)v_1$ is in V_w. After integrating over u_2, the inner integral does not depend on the choice of u_1 and is indeed a function of v_1 alone, which gives a meaning to the integral.

Proof: According to the previous lemma, we can write every $v \in V_w$ in the form $v = u_1\theta(v_1)v_1$ with $v_1 \in M \cap {}^\theta N$ and $u_1 \in N \cap {}^\theta N$ such that $u_1\theta(u_1)^{-1}$ is the commutator of v_1 and $\theta_1(v_1)$. Furthermore, this decomposition is compatible with the product in the sense that

$$(u_1\theta(v_1)v_1)(u_2\theta(v_2)v_2) = u_3\theta(v_1v_2)(v_1v_2),$$

as follows from the fact that the commutator of an element of $M \cap {}^\theta N$ and an element of ${}^\theta M \cap N$ is contained in $N \cap \theta(N)$. Moreover, N_w is a normal subgroup of V_w. The lemma follows from these observations.

There is a similar decomposition for the invariant measure on the quotient $V_w(F)\backslash V_w(F_{\mathbb{A}})$. We are now ready to state the main result of this section:

PROPOSITION 6.1: *Suppose ϕ is cuspidal on M^1. The integral*

$$\int_{T'_w(F_{\mathbb{A}})\backslash U(F_{\mathbb{A}})} du \int_{T_w(F)\backslash T_w(F_{\mathbb{A}})} \phi(t\eta_w u)dt \tag{64}$$

is zero unless w normalizes M. It is then equal to

$$\int_{T'_w(F_{\mathbb{A}})\backslash U(F_{\mathbb{A}})} du \int_{M_w(F)\backslash M_w(F_{\mathbb{A}})} \phi(t\eta_w u)\delta(t)dt, \tag{65}$$

where δ is the module of the group T_w.

Proof: We look at the inner integral; it has the form

$$\int_{T_w(F)\backslash T_w(F_{\mathbb{A}})} \phi(tg)dt.$$

It can be computed as

$$\int_{M_w(F)\backslash M_w(F_{\mathbb{A}})} \delta(m)dm \int_{V_w(F)\backslash V_w(F_{\mathbb{A}})} \phi(vmg)dv.$$

In turn, we look at the most inner integral:

$$\int_{V_w(F)\backslash V_w(F_{\mathbb{A}})} \phi(vmg)dv$$

and use the previous lemma. Since ϕ is invariant under $N(E_{\mathbb{A}})$, the integral over u_2 disappears and we are left with

$$\int_{V_w(F)\backslash V_w(F_{\mathbb{A}})} \phi(vmg)dv = \int_{M\cap^\theta N(E)\backslash M\cap^\theta N(E_{\mathbb{A}})} \phi(v_1g)dv_1.$$

Now suppose ϕ is cuspidal on M^1. Then the above integral is 0 unless $M \cap {}^\theta N = \{e\}$, that is, the parabolic subgroup $M \cap {}^\theta P$ of M is equal to M. This is equivalent to $M \cap {}^\theta M = M$ or $M \subseteq {}^\theta M$. Since ${}^\theta M$ is actually a conjugate of M, this is equivalent to $M = {}^\theta M$ or $wMw^{-1} = M$. This concludes the proof of the first assertion of the proposition. The second assertion follows.

Remark: Mutatis mutandis, the above argument applies to any symmetric space. Results of this type are stated in [JR] and [FJ], [yF2] and [yF3]. The discussion of the corresponding result in [yF2], although incomplete, is suggestive and we have benefited from it.

7. Integral of truncated Eisenstein series: II

From now on we take $n = 3$.

7.1 MAXIMAL PARABOLIC SUBGROUPS. We now consider the parabolic P_1 of type $(2,1)$ and the parabolic P_2 of type $(1,2)$. We set $\hat{\Delta}_i = \{\omega_i\}$. Then $\Omega(\mathfrak{a}_1, \mathfrak{a}_2)$ contains only one element s. We consider an Eisenstein series for P_1 say which is constructed from a cuspidal representation π of M_1^1.

PROPOSITION 7.1: *If T is sufficiently regular, the integral*

$$\int_{U(F)\backslash U(F_{\mathbb{A}})} \Lambda^T E(u, \phi, \pi_\zeta) du$$

does not depend on T.

Proof: By analytic continuation, it suffices to prove this when we can estimate the dominating series of the truncated Eisenstein series. Then the truncated Eisenstein series can be written as a sum of two terms

$$\sum_{P_1(E)\backslash G(E)} \theta_1(\gamma g) + \sum_{P_2(E)\backslash G(E)} \theta_2(\gamma g)$$

where θ_i is invariant under $N_i(E_{\mathbb{A}})$ and cuspidal on M_i. More precisely:

$$\theta_1(x) = \phi_2(\zeta_R, H_0(x) - T) e^{\langle \zeta + \rho_1, H_0(x) \rangle} \phi(x), \tag{66}$$

$$\theta_2(x) = \epsilon_2(s\zeta_R)\phi_2(s\zeta_R, H_0(x) - T) e^{\langle s\zeta + \rho_2, H_0(x) \rangle} M(s, \pi_\zeta)\phi(x). \tag{67}$$

We recall that

$$\phi_2(\zeta_R, H_0(x) - T) = 1$$

if and only if

$$(\omega_1, H_0(x)) \le \omega_1(T).$$

and

$$\phi_2(s\zeta_R, H_0(x) - T) = 1$$

if and only if

$$(\omega_2, H_0(x)) > \omega_2(T).$$

As observed before, we can compute formally. We consider a set of representatives for the orbits of $P_i(E)$ on the set of Hermitian matrices equivalent to σ. We can choose for representatives matrices of the form $\xi = w\alpha$, with α diagonal and w a matrix of permutation. For each such matrix ξ we choose η_ξ such that

$$\eta_\xi \sigma {}^t\overline{\eta}_\xi = \xi. \tag{68}$$

Then the integral of $\sum \theta_i(\gamma u)$ can be written as

$$\sum_\xi \int_{\eta_\xi^{-1} P_i(E)\eta_\xi \cap U(F)\backslash U(F_{\mathbb{A}})} \theta_i(\eta_\xi u) du. \tag{69}$$

By the previous section, the cuspidality implies that the integral is zero unless w normalizes M_i. Here this means that w is in M_i. Thus we need only to consider $\xi \in M_i$. Now the element

$$y = \eta_\xi u$$

verifies

$$y\sigma {}^t\overline{y} = \xi.$$

We now appeal to a lemma:

LEMMA 7.1: *Fix i. Let ξ be in M_i. If g is any element such that*

$$g\sigma {}^t\overline{g} = \xi,$$

then

$$\omega_i(H(g)) \le 0.$$

Proof: Using the decomposition $G = G^1 A_G$, we see that we may assume that $g \in G^1$. Suppose $i = 1$. We have

$$\exp(\omega_1, H(g)) = \|e_3 g\|^{-c},$$

for some positive constant c. Now

$$g\sigma = \xi^t\overline{g}^{-1}.$$

Since σ is in K and ξ is in M_1 we get

$$\|e_3 g\| = \|e_3 g\sigma\| = \|e_3 {}^t\overline{g}^{-1}\|.$$

On the other hand, writing $g \in G_1$ as $g = bk$ where b is a triangular matrix with diagonal entries b_i, we have

$$\|e_3 g\| = \|(0,0,1)g\| = |b_3|,$$

$$\|e_3 {}^t\overline{g}^{-1}\| = \|(*,*,\overline{b}_3^{-1})\| \geq |b_3|^{-1}.$$

Thus $|b_3| \geq 1$ and our conclusion follows in this case.

A similar argument applies to $i = 2$. We have

$$\exp(\omega_2, H(g)) = \|e_2 g \wedge e_3 g\|^{-c},$$

with $c > 0$. From

$$g\sigma = \xi^t\overline{g}^{-1}$$

we get

$$\|e_2 g \wedge e_3 g\| = \|e_3 {}^t\overline{g}^{-1} \wedge e_3 {}^t\overline{g}^{-1}\|.$$

Writing $g = bk$ as before, we get

$$\|e_2 g \wedge e_3 g\| = |b_2 b_3|,$$

$$\|e_3 {}^t\overline{g}^{-1} \wedge e_3 {}^t\overline{g}^{-1}\| = \|(*,*,\overline{b}_2^{-1}\overline{b}_3^{-1})\| \geq |b_2 b_3|^{-1}.$$

Hence $|b_2 b_3| \geq 1$ and our conclusion follows as before. ∎

Thus if $\omega_i(T) > 0$ we find that the function $\theta_1(\eta_\xi u)$ is actually independent of T :

$$\theta_1(\eta_\xi u) = e^{(\zeta+\rho_1, H_0(\eta_\xi u))}\phi(\eta_\xi u),$$

while $\theta_2(\eta_\xi u) = 0$. Our conclusion follows. ∎

We set

$$\Xi(\phi, \pi_\zeta) = \sum_\xi \int e^{(\zeta+\rho_1, H(\eta_\xi u))}\phi(\eta_\xi u)du. \tag{70}$$

Note that there are infinitely many terms and that the inner integral factors trough an integral of the form

$$\int \phi(v\eta_\xi u)dv$$

taken over the adelic quotient of $U_\xi \cap M_1$. We recall that

$$M_1 \simeq \mathrm{GL}(2) \times \mathrm{GL}(1).$$

In particular, the integral is zero unless in the representation $\pi = \pi_1 \otimes \pi_2$, the factor π_1 is distinguished with respect to a unitary group in GL(2) (not necessarily quasi-split) and the character π_2 distinguished with respect to the unitary group in one variable U_1. In particular, π_1 and π_2 must be quadratic base change. Each term in (70) is a linear form invariant under $U(\mathbb{A})$. Moreover

$$\int \Lambda^T E(u, \phi, \pi_\zeta)du = \Xi(\phi, \pi_\zeta). \tag{71}$$

Thus the right hand side is actually a meromorphic function of ζ with singularities contained in those of the Eisenstein series. It may be that each term in (70) has an analytic continuation as well but we will not investigate the matter further.

7.2 MINIMAL PARABOLIC SUBGROUP. We now discuss the integral of the truncated Eisenstein series attached to P_0.

LEMMA 7.2: *A set of representative for the orbits of $P_0(E)$ on the set of Hermitian matrices equivalent to σ consists of the matrices*

$$s_1, s_2, s_0 = \sigma$$

and diagonal matrices of the form

$$\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \alpha_3),$$

where $-\alpha_1\alpha_2\alpha_3$ is a norm and the α_i are in $F^\times$ and taken modulo norms.

The truncated Eisenstein series is written as

$$\Lambda^T E(g, \phi, \zeta) = \sum_{s \in \Omega(\mathfrak{a}_o)} \sum_{\delta \in P_0(E)\backslash G_0(E)} \theta_s(\delta g), \tag{72}$$

where

$$\theta_s(g) = \epsilon_2(s\zeta_R)\phi_2(s\zeta_R, H(x) - T)e^{(s\zeta+\rho, H(x))}(M(s,\zeta)\phi)(x). \tag{73}$$

We assume that $(\zeta_R, \alpha^\vee) > m(\rho_0, \alpha^\vee)$ for a suitable integer m. Then we can compute the integral of the Eisenstein series formally because the majorizing series is integrable over U.

The integral of the Eisenstein series is now

$$\sum_\xi \sum_s \int_{U(F)\cap\eta_\xi^{-1}P_0(E)\eta_\xi\backslash U(F_{\mathbb{A}})} \theta_s(\eta_\xi u)du, \tag{74}$$

where we sum over all elements ξ of the previous proposition and we choose η_ξ such that the relation (68) is satisfied. We first consider the subsum corresponding to the diagonal matrices.

PROPOSITION 7.2: *The sum*

$$\sum_\alpha \sum_s \int_{U(F)\cap\eta_\alpha^{-1}P_0(E)\eta_\alpha\backslash U(F_{\mathbb{A}})} \theta_s(\eta_\alpha u)du$$

is actually independent of T.

Proof: Note that if $y = \eta_\alpha u$ then

$$y\sigma{}^t\overline{y} = \alpha.$$

But for such an element we have the following result:

LEMMA 7.3: *Suppose that*

$$g\sigma{}^t\overline{g} = \alpha.$$

Then for $i = 1, 2$ *we have* $\omega_i(H(g)) \le 0$.

Indeed we have

$$g\sigma = \alpha{}^t\overline{g}^{-1}$$

and thus

$$(\omega_i, H(g)) = (\omega_i, H(g\sigma)) = (\omega_i, H({}^t\overline{g}^{-1}))$$

and one concludes as before. ∎

Thus in the expression for $\theta_s(\eta_\alpha u)$ the characteristic function ϕ_2 is zero unless $s = 1$ in which case it is one. This proves our contention. In particular the contribution of these terms can be written as

$$\Xi(\phi, \pi_\zeta) = \sum_\alpha \int e^{(\zeta+\rho_0, H(\eta_\alpha u))}\phi(\eta_\alpha u)du. \tag{75}$$

Once more the sum is infinite and the inner integral factors through an integral over $U_\alpha(F)\cap M_0(F)\backslash U_\alpha\cap M_0(F_{\mathbb{A}})$. Thus the integral is zero unless $\pi = \pi_1\otimes\pi_2\otimes\pi_3$ is distinguished with respect to the group $U_\alpha \cap M_0$; in turn, this intersection is the product of three copies of the unitary group in one varaible U_1. It will be convenient to introduce the notation

$$\delta(\pi_i) = \int_{U_1(F)\backslash U_1(F_{\mathbb{A}})} \pi_i(u)du.$$

We take the volume of $U_1(F)\backslash U_1(F_{\mathbb{A}})$ to be one. Thus this is 0 unless the character π_i is distinguished (i.e is a base change) in which case it is 1. Thus $\Xi = 0$ unless

$$\delta(\pi_1) = \delta(\pi_2) = \delta(\pi_3) = 1.$$

Again (75) is a linear form invariant under $U(\mathbb{A})$.

We will also set

$$\delta(\pi_i, \pi_j) = \int_{E^\times\backslash E^{\times,1}} \pi_i(a)\pi_j(\overline{a})^{-1}da.$$

Again, we take the volume of $E^\times\backslash E^{\times,1}$ to be one. We also use the exact sequence

$$1 \longrightarrow E^\times\backslash E^{\times,1} \longrightarrow E^\times\backslash E^\times_{\mathbb{A}} \xrightarrow{\log|.|} \mathbb{R} \longrightarrow 0$$

to define a measure on the idèle-class group of E. The integral is zero unless

$$\pi_i(a) = \pi_j(\overline{a}),$$

in which case it is 1.

Note that $\pi_1\pi_2\pi_3 = \omega$ and the character ω is itself distinguished. Thus $\delta(\pi_1) = \delta(\pi_2) = 1$, say, implies in fact $\delta(\pi_3) = 1$. Likewise, the relation $\delta(\pi_1, \pi_2) = 1$ implies in fact $\delta(\pi_3) = 1$. Finally if $\delta(\pi_1) = \delta(\pi_2) = 1$ then the relation $\delta(\pi_1, \pi_2) = 1$ is equivalent to $\pi_1 = \pi_2$.

Now we give the contribution of the remaining ξ.

PROPOSITION 7.3: *Assume $T_1 = \omega_1(T) = \omega_2(T)$ and write $\zeta = \zeta_1\omega_1 + \zeta_2\omega_2$. For $\xi = \sigma$.we can take $\eta_\xi = 1$ and the contribution is:*

$$\delta(\pi_1, \pi_3)\delta(\pi_2)\frac{e^{(\zeta_1+\zeta_2)T_1}}{\zeta_1+\zeta_2} \quad\times\quad \int \phi(k)dk \tag{76}$$

$$+\delta(\pi_1, \pi_3)\delta(\pi_2)\frac{e^{-(\zeta_1+\zeta_2)T_1}}{\zeta_1+\zeta_2} \quad\times\quad \int M(s_0, \pi_\zeta)\phi(k)dk \tag{77}$$

where the integral is over $K \cap U(F_{\mathbb{A}})$. For $\xi = s_1$ we can take $\eta_\xi = s_2$ and the contribution is

$$(78)\qquad \frac{e^{\zeta_1 T_1}}{\zeta_1}\delta(\pi_1,\pi_2)\delta(\pi_3) \quad\times\quad \int M(s_2,\zeta_2\omega_2)\phi(k)dk$$

$$(79)\qquad -\frac{e^{-\zeta_1 T_1}}{\zeta_1}\delta(\pi_1,\pi_2)\delta(\pi_3) \quad\times\quad \int M(s_2,(\zeta_1+\zeta_2)\omega_2)M(s_1,\zeta)\phi(k)dk$$

$$(80)\qquad -\frac{e^{-(\zeta_1+\zeta_2)T_1}}{(\zeta_1+\zeta_2)}\delta(\pi_1,\pi_3)\delta(\pi_2) \quad\times\quad \int M(s_2,\zeta_1\omega_2)M(s_1s_2,\zeta)\phi(k)dk.$$

Finally, for $\xi = s_2$ we can take $\eta_\xi = s_1$ and the contribution is

$$(81)\qquad \frac{e^{\zeta_2 T_1}}{\zeta_2}\delta(\pi_2,\pi_3)\delta(\pi_1) \quad\times\quad \int M(s_1,\zeta_1\omega_1)\phi(k)dk$$

$$(82)\qquad -\frac{e^{-\zeta_2 T_1}}{\zeta_2}\delta(\pi_2,\pi_3)\delta(\pi_1) \quad\times\quad \int M(s_1,(\zeta_1+\zeta_2)\omega_1)M(s_2,\zeta)\phi(k)dk$$

$$(83)\qquad -\frac{e^{-(\zeta_1+\zeta_2)T_1}}{(\zeta_1+\zeta_2)}\delta(\pi_1,\pi_3)\delta(\pi_2) \quad\times\quad \int M(s_1,\zeta_2\omega_1)M(s_2s_1,\zeta)\phi(k)dk.$$

The integral

$$\int \Lambda^T E(u,\phi,\pi_\zeta)du$$

is equal to the sum of the previous terms and

$$(84)\qquad \delta(\pi_1)\delta(\pi_2)\delta(\pi_3)\Xi(\phi,\pi_\zeta).$$

For clarity, we have suppressed the representation π from the notation in the intertwining operators. Thus in (78) the intertwining operator should be written as

$$M(s_2,\pi_{\zeta_2\omega_2}).$$

Likewise in (79) the operator on the left should be written as

$$M(s_2,\pi'_{(\zeta_1+\zeta_2)\omega_2})$$

where $\pi' = s_1\pi = \pi_2 \otimes \pi_1 \otimes \pi_3$.

Proof: Consider first the case of σ. Since $\eta_\sigma = 1$ the contribution of σ can be written as

$$\sum_s \int \theta_s(u)du.$$

We use the Iwasawa decomposition in U induced by the Iwasawa decomposition of G to define a Haar measure. If

$$u = \begin{pmatrix} 1 & x & z + x\overline{x}/2 \\ 0 & 1 & -\overline{x} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \overline{a}_1^{-1} \end{pmatrix} k, \tag{85}$$

with $z + \overline{z} = 0$ and $m\overline{m} = 1$, then

$$du = dxdzd^{\times}a_1dkdm|a_1|^{-2}.$$

We also write this decomposition $u = nak$ where

$$a = \mathrm{diag}(a_1, m, \overline{a_1}^{-1}).$$

Then

$$e^{2\rho_U(H_U(a))} = |a_1|^2 = e^{\rho_0(H(a))}. \tag{86}$$

Also

$$\log|a_1| = \omega_1(H(u)) = \omega_2(H(u)) = t. \tag{87}$$

Recall we assume that $T_1 = \omega_1(T) = \omega_2(T)$. Referring to the table after Proposition 5.1, we see that $\phi_2(s\zeta_R, H(u) - T) = 0$ except for $s = e$ and $s = s_0$. Indeed, for $s = s_1$ for instance, the relation $\phi_2(s\zeta_R, H(u) - T) \neq 0$ implies

$$T_1 < \omega_1(H(u)) = \omega_2(H(u)) \leq T_1$$

and these two inequalities are not compatible.

After integrating over $M_0^1 \cap U$ we get for the contribution of σ:

$$\delta(\pi_1, \pi_3)\delta(\pi_2) \int_{-\infty}^{T_1} e^{(\zeta_1+\zeta_2)t} dt \int \phi(k)dk$$

$$+\delta(\pi_1, \pi_3)\delta(\pi_2) \int_{T_1}^{+\infty} e^{-(\zeta_1+\zeta_2)t} dt \int M(s_0, \zeta)\phi(k)dk$$

which yields at once the required result.

We pass to the contribution of s_1. It is equal to

$$\sum_s \int du \int \theta_s(gs_2u)dg. \tag{88}$$

The inner integral is over the intersection

$$T = P_0 \cap s_2Us_2^{-1}.$$

This can described as the group of matrices of the form

$$g = \begin{pmatrix} 1 & z & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a_1 & 0 & 0 \\ 0 & \overline{a}_1^{-1} & 0 \\ 0 & 0 & m \end{pmatrix}, \tag{89}$$

with $z + \overline{z} = 0$ and $m\overline{m} = 1$. We remark that $s_2 T s_2^{-1}$ is a subgroup of $P_0 \cap U$. We take for left Haar measure on T the one defined by

$$dg = dz dm d^\times a_1 |a_1|^{-1}.$$

Thus we have for $g \in T$

$$\omega_2(H(g)) = 0,$$

$$\rho_0(H(g)) = \omega_1(H(g)) = \log |a_1| = t,$$

$$H(gs_2u) = H(g) + H(s_2u),$$

$$\omega_1(H(gs_2u)) = \omega_1(H(g)) + \omega_1(H(s_2u)).$$

Recall

$$\theta_s(gs_2u) = \epsilon_2(s\zeta_R)\phi_2(s\zeta_R, H(gs_2u) - T)e^{(s\zeta+\rho_0, H(gs_2u))}M(s,\zeta)\phi(gs_2u).$$

Since $y = gs_2u$ verifies

$$y\sigma = s_1 {}^t\overline{y}^{-1}$$

we have, as before, $\omega_2(H(gs_1u)) \leq 0$. Referring again to the table after Proposition 5.1, it follows that

$$\phi_2(s\zeta_R, H(gs_2h) - T) = 0$$

unless $s = e, s_1, s_1s_2$. Moreover, for these terms one of the two inequalities which define ϕ_2 is vacuous. Consider thus the term $s = e$. Then

$$\phi_2(\zeta_R, H(gs_2u) - T) \neq 0$$

if and only if

$$\omega_1(H(g)) \leq T_1 - \omega_1(H(s_2u)).$$

Integrating over the group T we get

$$\delta(\pi_1, \pi_2)\delta(\pi_3) \int du \int_{-\infty}^{T_1 - \omega_1(H(s_2u))} e^{t\zeta_1} dt e^{(\zeta_1\omega_1 + \zeta_2\omega_2 + \rho_0, H(s_2u))}\phi(s_2u).$$

This becomes

$$\delta(\pi_1,\pi_2)\delta(\pi_3)\frac{e^{\zeta_1 T_1}}{\zeta_1}\int e^{(\zeta_2\omega_2+\rho_0,H(s_2u))}\phi(s_2u)du.$$

To continue we remark that $s_2Ts_2^{-1}$ is the group of matrices of the form

$$\begin{pmatrix} a_1 & 0 & z \\ 0 & m & 0 \\ 0 & 0 & \overline{a}_1^{-1} \end{pmatrix}.$$

Thus, to integrate over U modulo this group as we must, we can use the following system of coordinates:

$$u=\begin{pmatrix} 1 & -\overline{x} & x\overline{x}/2 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}k$$

and then

$$du = dxdk.$$

In view of the invariance of ϕ under N_0 we see that the inner integral becomes

$$\delta(\pi_1,\pi_2)\delta(\pi_3)\frac{e^{\zeta_1 T_1}}{\zeta_1}\int e^{(\zeta_2\omega_2+\rho_0,H(s_2nk))}\phi(s_2nk)dxdk$$

where

$$n=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & x \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus the integral in n can be interpreted as an intertwining operator and we obtain the first term of the contribution of s_1. The other terms are obtained in a similar way.

Note that all the terms in the previous proposition, with the possible exception of Ξ, are meromorphic functions of ζ. It follows that the same is true of Ξ as well. Again, it may be that each term in (75) is meromorphic as well.

7.3 SINGULARITIES. We now analyze the singularities of each term in the previous proposition. If $\delta(\pi_1,\pi_3)=\delta(\pi_1,\pi_2)=\delta(\pi_2,\pi_3)=0$ the integral is zero unless

$$\delta(\pi_1)=\delta(\pi_2)=\delta(\pi_3)=1.$$

It then reduces to the term $\Xi(\phi,\pi_\zeta)$ whose singularities are thus contained in the singularities of the Eisenstein series.

Now assume that

$$\delta(\pi_1, \pi_3)\delta(\pi_2) = 1.$$

Consider the sums of all terms with a singularity on the line $\zeta_1 + \zeta_2 = 0$: (76), (77), (80), (83). We first claim that the sum of (77) and (83) has no singularity along the line $\zeta_1 + \zeta_2 = 0$. Indeed, on this line, we have $\zeta_2\omega_1 = s_2 s_1 \zeta$ and thus

$$M(s_0, \zeta) = M(s_1, s_2 s_1 \zeta) M(s_2 s_1, \zeta) = M(s_1, \zeta_2 \omega_1) M(s_2 s_1, \zeta).$$

Thus the singularities do cancel. Consider the sum of (76) and (80). Suppose that $\pi_1 \neq \pi_3$. This is equivalent to $\delta(\pi_1) = \delta(\pi_3) = 0$. Then all the other terms in the expression for the integral of the truncated Eisenstein series vanish (including Ξ). Thus the sum of these two terms must have no singularity on $i\mathfrak{a}^*$. In particular, it follows that

$$\int \phi(k) dk = \int M(s_2, \zeta_1 \omega_2) M(s_1 s_2, \zeta) \phi(k) dk$$

along the line $\zeta_1 + \zeta_2 = 0$. Again we have

$$M(s_0, \zeta) = M(s_2, s_1 s_2 \zeta) M(s_1 s_2, \zeta).$$

Along the line $\zeta_1 + \zeta_2 = 0$ we have $s_1 s_2 \zeta = \zeta_1 \omega_2$. Thus the above identity can be written

$$\int \phi(k) dk = \int M(s_0, \zeta) \phi(k) dk. \tag{90}$$

Equivalently, we can write the sum of (76), (77), (80), (83) in the form

$$\frac{e^{(\zeta_1+\zeta_2)T_1} - e^{-(\zeta_1+\zeta_2)T_1}}{\zeta_1 + \zeta_2} \int \phi(k) dk + e^{-(\zeta_1+\zeta_2)T_1} F(\zeta) \tag{91}$$

where F is a smooth function (of slow growth) on $i\mathfrak{a}^*$.

Now suppose that $\pi_1 = \pi_3$. Thus $\delta(\pi_1) = \delta(\pi_3) = 1$. We claim that

$$M(s_0, \zeta) = -1$$

along the line $\zeta_1 + \zeta_2 = 0$. Indeed,

$$M(s_0, \zeta) = M(s_1, s_2 s_1 \zeta) M(s_2, s_1 \zeta) M(s_1, \zeta).$$

However, $s_2 s_1 \zeta = s_1 \zeta$ since $\zeta_1 + \zeta_2 = 0$ and $s_2 s_1 \pi = s_1 \pi$ since $\pi_1 = \pi_3$. It follows from the properties of the intertwining operator on GL(2) that

$$M(s_2, s_1 \zeta) = -1$$

and

$$M(s_0,\zeta) = -M(s_1, s_2s_1\zeta)M(s_1,\zeta).$$

Using again the fact that $s_2s_1\zeta = s_1\zeta$ and $s_2s_1\pi = s_1\pi$ we find this is

$$= -M(s_1, s_1\zeta)M(s_1,\zeta) = -M(s_1^2,\zeta) = -1.$$

On the other hand, from the functional equation of the Eisenstein series, we get

$$E(x, M(s_0,\zeta)\phi,\zeta) = E(x,\phi,\zeta)$$

and it follows that the Eisenstein series vanishes along the line $\zeta_1 + \zeta_2 = 0$. Note that in this case the singularities of (76) and (80) do not cancel but double.

We now assume that

$$\delta(\pi_1,\pi_2)\delta(\pi_3) = 1$$

and consider the sum of the terms which have a singularity along the line $\zeta_1 = 0$: (78) and (79). Assume first $\pi_1 \neq \pi_2$. Then $\delta(\pi_1) = \delta(\pi_2) = 0$ and all the other terms vanish including Ξ and we conclude that the residues of the two terms along the line $\zeta_1 = 0$ cancel, that is,

$$\int M(s_2,\zeta_2\omega_2)\phi(k)dk = \int M(s_2,\zeta_2\omega_2)M(s_1,\zeta_2\omega_2)\phi(k)dk.$$

Equivalently, we can write the sum of (78) and (79) in the form

$$\frac{e^{\zeta_1 T_1} - e^{-\zeta_1 T_1}}{\zeta_1}\int M(s_2,\zeta_2\omega_2)\phi(k)dk + e^{-\zeta_1 T_1}F(\zeta) \tag{92}$$

where F is a smooth function (of slow growth) on $i\mathfrak{a}^*$.

If, on the contrary, $\pi_1 = \pi_2$, then $\delta(\pi_1) = \delta(\pi_2) = 1$ and on the line $\zeta_1 = 0$ we have $s_1\zeta = \zeta$ and $s_1\pi = \pi$. It follows that $M(s_1,\zeta) = -1$ and again the Eisenstein series vanishes on this line. Again in this case the singularities do not cancel but double.

A similar discussion applies to the case where

$$\delta(\pi_2,\pi_3)\delta(\pi_1) = 1$$

and the remaining terms.

We summarize our discussion as follows.

PROPOSITION 7.4:

(i) *Suppose that*

$$\delta(\pi_1) = \delta(\pi_2) = \delta(\pi_3) = 1$$

but not two of the characters π_i are equal. Then the integral of the truncated Eisenstein series reduces to Ξ and in particular has no singular line on $i\mathfrak{a}^*$.

(ii) *Suppose that*

$$\delta(\pi_1) = \delta(\pi_2) = \delta(\pi_3) = 1$$

and at least two of the characters are equal. Then the product of each term in the previous proposition by

$$\overline{W(\phi_2, \pi_\zeta)} \tag{93}$$

is a smooth function of slow growth on $i\mathfrak{a}^*$.

(iii) *Suppose that $\delta(\pi_i) = 1$ but $\delta(\pi_j) = \delta(\pi_k) = 0$. Then $\Xi = 0$. All terms vanish unless*

$$\delta(\pi_j, \pi_k) = 1.$$

Then the integral of the truncated Eisenstein series can be written in the form

$$\frac{e^{(\zeta,\check{\alpha})T_1} - e^{-(\zeta,\check{\alpha})T_1}}{(\zeta,\check{\alpha})} A(\zeta) + e^{-(\zeta,\check{\alpha})T_1} B(\zeta)$$

where A and B are smooth functions of slow growth on $i\mathfrak{a}^*$. Here $\check{\alpha}$ is the coroot attached to the root $\alpha_1 + \alpha_2$ if $\{j,k\} = \{1,3\}$, the root α_1 if $\{j,k\} = \{1,2\}$, the root α_2 if $\{j,k\} = \{2,3\}$.

In all other case, the integral of the truncated Eisenstein series is 0.

Proof: By a smooth function of slow growth on $i\mathfrak{a}^*$ we mean a smooth function whose derivatives of all orders are at most of polynomial growth. We have seen that the matrix coefficients of the intertwining operator are smooth functions of slow growth. Thus the only point which remains to be verified is (ii). Assume that

$$\delta(\pi_1) = \delta(\pi_2) = \delta(\pi_3) = 1$$

and say

$$\pi_1 = \pi_2 = \pi_3.$$

Consider the terms other than Ξ. Each term has only one singular line which intersects $i\mathfrak{a}^*$. On such a line the Eisenstein series vanishes. Thus our assertion is satisfied for each term different of Ξ. Since the Eisenstein series itself has no singularity on $i\mathfrak{a}^*$, the assertion follows for the term Ξ as well. ∎

8. Spectral contribution

In this section we obtain the spectral contribution of each parabolic subgroup.

8.1 MAXIMAL PARABOLIC SUBGROUPS. We now consider the spectral contribution of the parabolic subgroups P_1 of type $(2,1)$ and P_2 of type $(1,2)$. Let χ be a cuspidal datum for these parabolic subgroups. Thus χ can be viewed as a cuspidal automorphic representation of $\mathrm{GL}(2) \times \mathrm{GL}(1)$ and defines representations π of M_i^1, $i = 1, 2$. Recall the corresponding Eisenstein series gives no residue.

PROPOSITION 8.1: *We have*

$$\int K_\chi(u,n)du\bar{\theta}(n)dn = \sum_{P_i} \int_{i\mathfrak{a}^*} \sum_\phi \Xi(I_{P_i}(\pi_\zeta, f)\phi, \pi_\zeta)\overline{W(\phi, \pi_\zeta)}d|\zeta|.$$

Furthermore, the sum

$$\sum_\chi \sum_{P_i} \int \left| \sum_\phi \Xi(I_{P_i}(\pi_\zeta, f)\phi, \pi_\zeta)\overline{W(\phi, \pi_\zeta)} \right| d|\zeta|$$

is finite.

Proof: It is understood that in these formulas, the representation π is determined by χ in the sense that (P_i, M_i, π) belongs to the class χ. We first prove the last assertion. As we have seen, there is T_0 (independent of f and χ) such that

$$\Xi(\phi, \pi_\zeta) = \int \Lambda^{T_0} E(u, \phi, \pi_\zeta)du.$$

Thus the expression we have to estimate is

$$\sum_\chi \sum_{P_i} \int \left| \sum_\phi \int \Lambda^{T_0} E(u, I_{P_i}(\pi_\zeta, f)\phi, \pi_\zeta)du\overline{W(\phi, \pi_\zeta)} \right| d|\zeta|.$$

Our assertion follows from the basic majorization (23).

The first assertion follows similarly from Proposition 2.5.

8.2 The minimal parabolic subgroup. We pass to the contribution of the minimal parabolic subgroup $P_0 = M_0N_0$. A cuspidal datum is then a triple of characters of G_m^1, $\chi = \chi_1 \otimes \chi_2 \otimes \chi_3$, defined up to a permutation. In addition $\chi_1\chi_2\chi_3 = \omega$, where ω is fixed and a base change. By Proposition 2.5, if T is sufficiently regular,

$$\int K_\chi(u,n)\theta(n)dudn = \int \Lambda^T K_\chi(u,n)\theta(n)dudn$$

$$= n(A_0)^{-1}\sum_\pi\sum_\phi\int\left(\int \Lambda^T E(u, I_0(\pi_\zeta, f)\phi, \zeta)du\right)\overline{W(\phi,\pi_\zeta)}d|\zeta|.$$

The sum is over all permutations $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3$ of the given triple χ. We have written $I_0(\pi_\zeta)$ for the induced representation $I_{P_0}(\pi_\zeta)$. We recall that $n(A_0) = 6$. We may further assume that $\omega_1(T) = \omega_2(T) = T_1$ as in the previous section.

Proposition 8.2: *Assume that $f = f_1 * f_2^*$ where the functions f_i are K-finite and $f_2^*(g) = \overline{f_2(g^{-1})}$. Suppose that $\delta(\chi_i) = 1$ for $i = 1, 2, 3$. Then*

$$\int K_\chi(u,n)du\theta(n)dn = \frac{1}{6}\sum_\pi\int\sum_\phi \Xi(I_0(\pi_\zeta, f_1)\phi, \pi_\zeta)\overline{W(I(\pi_\zeta, f_2)\phi, \pi_\zeta)}\,d|\zeta|.$$

Furthermore, if we sum over all such triples χ then

$$\sum_\chi\sum_\pi\int\left|\sum_\phi \Xi(I_0(\pi_\zeta, f_1)\phi, \pi_\zeta)\overline{W(I(\pi_\zeta, f_2)\phi, \pi_\zeta)}\right|\,d|\zeta|$$

is finite.

Proof: Suppose that not two of the characters χ_j are equal. Then

$$\delta(\chi_i, \chi_j) = 0$$

for $i \neq j$. Thus the integral of the truncated Eisenstein series reduces to the term Ξ and we can argue as in the previous case. In particular, the contribution of these triples to the sum with the absolute values is finite. Now suppose that the three characters are equal so that $\chi_1^3 = \omega$. Then

$$\int\sum_\phi \Lambda^T E(u, I_0(\pi_\zeta, f)\phi, \pi_\zeta)du\,\overline{W(\phi,\pi_\zeta)}$$

is the sum of

$$\Xi(I_0(\pi_\zeta, f_1)\phi, \pi_\zeta)\overline{W(I_0(\pi_\zeta, f_2)\phi, \pi_\zeta)}$$

and 8 other terms. For instance,

$$e^{-(\zeta_1+\zeta_2)T_1}\frac{h(z)}{\zeta_1+\zeta_2}$$

where

$$h(z)=\int\sum_{\phi}M(s_0,\pi_\zeta)I_0(\pi_\zeta,f_1)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}.$$

On $i\mathfrak{a}^*$ the function $h(\zeta)$ is a Schwartz function. Since the Eisenstein series $E(y,\phi,\pi_\zeta)$ vanishes on the line $\zeta_1+\zeta_2=0$, so does h. Thus the quotient $h(z)/(\zeta_1+\zeta_2)$ is again a Schwartz function. In particular, it is integrable and its integral against the exponential factor tends to 0 as T_1 tends to infinity. We obtain then the first assertion of the proposition, plus the fact that, for a given triple χ, the sum

$$\sum_{\pi}\int\left|\sum_{\phi}\Xi(I_0(\pi_\zeta,f_1)\phi,\pi_\zeta)\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}\right|d|\zeta|$$

is finite. Since there are only finitely many triples $\chi=\chi_1\otimes\chi_1\otimes\chi_1$ for which $I_0(\pi_\zeta,f_1)\neq 0$ and $\chi_1^3=\omega$, their contribution to the infinite sum is finite.

We now consider the case where two of the characters are the same, say $\chi_1=\chi_2\neq\chi_3$. Thus $\delta(\chi_1,\chi_2)=1$ and $\delta(\chi_3)=1$. On the other hand $\delta(\chi_1,\chi_3)=\delta(\chi_2,\chi_3)=0$. We obtain again the first assertion of the proposition. We have to show that the contribution of these triples to the sum with the absolute values is finite. We consider for instance the contribution of the π of the form $\pi_i=\chi_i$. We can write

$$\Xi(I_0(\pi_\zeta,f_1)\phi,\pi_\zeta)\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}$$

$$=\int\Lambda^T E(u,I_\pi(f_1)\phi,\pi_\zeta)du\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}\tag{94}$$

$$-\frac{e^{\zeta_1T_1}}{\zeta_1}\int M(s_2,\zeta_2\omega_2)I_0(\pi_\zeta,f_1)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}\tag{95}$$

$$\frac{e^{-\zeta_1T_1}}{\zeta_1}\int M(s_2,(\zeta_1+\zeta_2)\omega_2)M(s_1,\zeta)I_0(\pi_\zeta,f)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}.$$

We have already observed many times that the expression

$$\sum_{\pi}\int\left|\sum_{\phi}\int\Lambda^T E(u,I_0(\pi_\zeta,f_1)\phi,\pi_\zeta)du\,\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}\right|d|\zeta|,$$

where we sum over all π, is finite. Thus it suffices to prove the analogous assertion for the two remaining terms. For the second term say, this amounts to proving that

$$(96)\quad \sum_{\pi}\int\left|\frac{1}{\zeta_1}\sum_{\phi}\int M(s_2,\zeta_2\omega_2)I_0(\pi_\zeta,f_1)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}\right|d|\zeta|$$

is finite; the sum is over all π of the above form. From Proposition 3.3, for f_2 fixed with a given K-type, we have a majorization of

$$\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}$$

and its derivatives. Since the function vanishes on $\zeta_1 = 0$ we get a majorization of the quotient of the form

$$\left|\frac{\overline{W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta)}}{\zeta_1}\right| \leq C(\chi)(1+\|\zeta\|^2)^{m(\chi)}\|\phi\|.$$

On the other hand, we have

$$\left|\int M(s_2,\zeta_2\omega_2)I_0(\pi_\zeta,f_1)\phi(k)dk\right| \leq \|I_0(\pi_\zeta,f_1)\|,$$

where the norm is the operator norm on the space of vectors with a given K-type. Thus the expression (96) is bounded by the dimension of the K-type times

$$\sum_{\pi}C(\chi)\int\|I_0(\pi_\zeta,f_1)\|(1+\|\zeta\|^2)^{m(\chi)}d|\zeta|.$$

In turn, the norm of the operator is bounded by the norm of a suitable matrix of the form

$$\left(\int_{A_0N_0} f_1(k_i^{-1}nak_j)dn\chi(a)e^{(H(a),\zeta-\rho)}da\right)$$

where k_i are points in K. It follows from abelian harmonic analysis that

$$\sum_{\pi}C(\chi)\int\left|\int f_1(k_i^{-1}nak_j)dn\chi(a)e^{(H(a),\zeta-\rho)}da\right|(1+\|\zeta\|^2)^{m(\chi)}d|\zeta|$$

is finite. Our assertion follows. ∎

We may further simplify the result of Proposition 8.2. To begin with, the expression in the Proposition may be written

$$\int\sum_{\phi\in\mathcal{B}_0(\pi)}\Xi(I_0(\pi_\zeta,f)\phi,\pi_\zeta)\overline{W(\phi,\pi_\zeta)}d|\zeta|.$$

We claim further that this expression does not depend on the choice of π in the class of χ. Indeed, there is nothing to prove if all the characters χ_j are equal. Suppose on the contrary that not two of the characters are equal. Then, for T suitably regular,

$$\Xi(\phi, \pi_\zeta) = \int \Lambda^T E(u, \phi, \pi_\zeta) du.$$

It follows from the functional equation of the Eisenstein series that

$$\Xi(\phi, \pi_\zeta) = \Xi(M(s, \pi_\zeta)\phi, s\pi_{s\zeta})$$

and

$$W(\phi, \pi_\zeta) = W(M(s, \pi_\zeta)\phi, s\pi_{s\zeta})$$

for any s. Thus

$$\int \sum_{\phi \in \mathcal{B}_0(\pi)} \Xi(I_0(\pi_\zeta, f)\phi, \pi_\zeta)\overline{W(\phi, \pi_\zeta)} d|\zeta|$$

$$= \int \sum_{\phi \in \mathcal{B}_0(\pi)} \Xi(I_0(s\pi_{s\zeta}, f)M(s, \pi_\zeta)\phi, s\pi_{s\zeta})\overline{W(M(s, \pi_\zeta)\phi, \pi_{s\zeta})} d|\zeta|$$

Since $M(s, \pi_\zeta)\phi$ is an orthonormal basis of $\mathcal{H}^0(s\pi)$ the inner sum can be rewritten as

$$\sum_{\phi \in \mathcal{B}_0(s\pi)} \Xi(I_0(s\pi_{s\zeta}, f)\phi, s\pi_{s\zeta})\overline{W(\phi, s\pi_{s\zeta})}.$$

When we integrate over ζ we may take $s\zeta$ for variable to obtain our assertion.

If two of the characters are equal let us set

$$O^T(\phi, \pi_\zeta) = \Xi(\phi, \pi_\zeta) - \int \Lambda^T E(u, \phi, \pi_\zeta) du.$$

As we have seen, the integral of

$$\sum_\phi O^T(I_0(\pi_\zeta, f)\phi, \pi_\zeta)\,\overline{W(\phi, \pi_\zeta)}$$

against $d|\zeta|$ tends to 0 as T_1 tends to infinity. Likewise, the integral of

$$= \sum_{\phi \in \mathcal{B}_0(s\pi)} O^T(I(s\pi_{s\zeta}, f)\phi, s\pi_{s\zeta})\overline{W(\phi, s\pi_{s\zeta})}$$

against $d|\zeta|$ tends to zero. It follows that the difference

$$\sum_{\phi \in \mathcal{B}_0(\pi)} \Xi(I_0(\pi_\zeta)\phi, \pi_\zeta)\overline{W(\phi, \pi_\zeta)} - \sum_{\phi \in \mathcal{B}_0(s\pi)} \Xi(I_0 s\pi_{s\zeta})\phi, s\pi_{s\zeta})\overline{W(\phi, s\pi_{s\zeta})}$$

has a zero integral against $d|\zeta|$. Thus we arrive at the same conclusion as before.

We now consider the case where two of the numbers $\delta(\chi_j)$ are zero. We fix a representative χ in the class. Say

$$\delta(\chi_1) = \delta(\chi_3) = 0.$$

Then the integral of the truncated Eisenstein series is zero unless

$$\delta(\chi_1, \chi_3) = 1, \quad \delta(\chi_2) = 1.$$

We have also

$$\chi_1 \neq \chi_3$$

and

$$\delta(\chi_1, \chi_2) = \delta(\chi_3, \chi_2) = 0.$$

Indeed if one of these relations is not satisfied then $\delta(\chi_1) = \delta(\chi_3) = 1$. At this point we must pay attention to the choice of the measures. The space $\mathfrak{a}$ has a metric invariant under $\Omega(\mathfrak{a})$; its dual has the corresponding metric. The Haar measure on $i\mathfrak{a}^*$ or a subspace is the one associated with the metric. In particular, we will denote by C the common length of the coroots.

PROPOSITION 8.3: *Under the above assumptions we have*

$$\int K_\chi(u,n)du\overline{\theta(n)}dn$$

(97)
$$= 2\pi C^{-1}$$

(98)
$$\times \int_{\zeta_1+\zeta_2=0} \sum_{\phi\in\mathcal{B}_{P_0}(\chi)} \int_{K_U} I_0(\chi_\zeta, f_1)\phi(k)dk\,\overline{W(I_0(\chi_\zeta, f_2)\phi, \chi_\zeta)}d|\zeta|.$$

Here $K_U = K \cap U(F_{\mathbb{A}})$. If we interchange χ_3 and χ_1 the above expression does not change. Furthermore the following sum, taken over all such triples χ, is finite:

$$\sum_\chi \int_{\zeta_1+\zeta_2=0} \left| \sum_\phi \int I_0(\chi_\zeta, f_1)\phi(k)dk\,\overline{W(I_0(\chi_\zeta, f_2)\phi, \zeta)} \right| d|\zeta|.$$

Proof: The proof of the last assertion follows again from the estimates on the Eisenstein series. To prove the first assertion we consider first the contribution of the representation $\pi = \chi$. It can be written as

$$\int \frac{e^{(\zeta_1+\zeta_2)T_1} - e^{-(\zeta_1+\zeta_2)T_1}}{(\zeta_1+\zeta_2)} \sum_\phi \int I_0(\chi_\zeta, f_1)\phi(k)dk\,\overline{W(I_0(\chi_\zeta, f_2)\phi, \chi_\zeta)}d|\zeta|$$
$$+ \int e^{-(\zeta_1+\zeta_2)T_1} F(\zeta)d|\zeta|$$

where F is a Schwartz function. The second term tends to 0 as T_1 tends to infinity. For the first term we apply the limit formula

$$\lim_{T\to+\infty}\int \frac{e^{iT<\lambda,\zeta>}-e^{-iT<\lambda,\zeta>}}{<\lambda,\zeta>}F(\zeta)d|\zeta| = 2i\pi\|\lambda\|^{-1}\int_{<\lambda,\zeta>=0}F(\zeta)d|\zeta|,$$

which is valid on any Euclidean space, for any Schwartz function F. We find this contribution is equal to (98) times $2\pi C^{-1}$.

Now we claim that the terms corresponding to $\pi = \chi_1\otimes\chi_3\otimes\chi_2$ actually give the same contribution. Indeed this contribution can be written as

$$\int \frac{e^{\zeta_1 T_1}-e^{-\zeta_1 T_1}}{\zeta_1}\sum_\phi\int M(s_2,\zeta_2\omega_2)I_0(\pi_\zeta,f_1)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi)}d|\zeta|$$

$$+\int e^{-\zeta_1 T_1}F(\zeta)d|\zeta|.$$

The second term tends to 0 and the first term tends to $2\pi C^{-1}$ times

$$\int_{\zeta_1=0}\sum_\phi\int M(s_2,\pi_\zeta)I_0(\pi_\zeta,f_1)\phi(k)dk\,\overline{W(I_0(\pi_\zeta,f_2)\phi)}d|\zeta|.$$

Now, by the functional equation of Eisenstein series we have

$$W(I_0(\pi_\zeta,f_2)\phi,\pi_\zeta) = W(I_0(s_2\pi_{s_2\zeta},f_2)M(s_2,\pi_\zeta)\phi,s_2\pi_{s_2\zeta}).$$

However, here $s_2\pi = \chi$. Hence our expression is also

$$\int_{\zeta_1=0}\sum_\phi\int I_0(\chi_{s_2\zeta},f_1)M(s_2,\pi_\zeta)\phi(k)dk\,\overline{W(I_0(\chi_{s_2\zeta},f_2)M(s_2,\pi_\zeta)\phi,\chi_{s_2\zeta})}d|\zeta|.$$

Consider the integrand, for a fixed ζ. Since the intertwining operator is unitary, $M(s_2,\pi_\zeta)\phi$ is an orthonormal basis of the space $\mathcal{H}_0^0(\chi)$. Any basis will give the same integrand. Thus our expression is equal to

$$\sum_\phi\int I_0(\chi_{s_2\zeta},f_1)\phi(k)dk\,\overline{W(I(\chi_{s_2\zeta},f_2)\phi,\chi_{s_2\zeta})}d|\zeta|.$$

Using $s_2\zeta$ for new variable of integration we arrive at (98).

The term corresponding to $\pi = \chi_2\otimes\chi_1\otimes\chi_3$ gives the same contribution. Exchanging χ_1 and χ_3 we now compare the contributions of the representations $\pi = \chi_1\otimes\chi_2\otimes\chi_3$ and $\pi' = \chi_3\otimes\chi_2\otimes\chi_1$. We recall that for $\phi\in\mathcal{B}_0(\pi)$ we have

$$\int\phi(k)dk = \int M(s_0,\pi_\zeta)\phi(k)dk$$

on the line $\zeta_1 + \zeta_2 = 0$. We have also

$$W(M(s_0, \pi_\zeta)\phi, s_0\pi_{s_0\zeta}) = W(\phi, \pi).$$

As before it follows that the contribution of π and π' are the same. Thus in fact all the permutations of χ give the same contribution. Since $n(A_0) = 6$ we are done. ∎

Under the assumptions of the proposition, consider a $\zeta \in i\mathfrak{a}^*$ such that $\zeta_1 + \zeta_2 = 0$. For $\phi \in \mathcal{H}_0^0(\chi)$ set

$$\phi_\zeta(g) = \phi(g)e^{(\zeta+\rho_0, H(g))}.$$

The subgroup $P_U = P_0 \cap U$ is a parabolic subgroup of U. Then, for $p \in P_U$,

$$\phi_\zeta(pg) = e^{(2\rho_{P_U}, H_{P_U}(g))}.$$

It follows that

$$\Xi(\phi, \pi_\zeta) = \int_{K_U} \phi_\zeta(k)dk = \int_{K_U} \phi(k)dk$$

is a linear form on the space of the induced representation $I_0(\pi_\zeta)$ which is invariant under $U(F_{\mathbb{A}})$. Thus we may write the result of the previous proposition as

$$\int_{\zeta_1+\zeta_2=0} \sum_{\phi\in\mathcal{B}_{P_0}(\chi)} \int_{K_U} \Xi(\chi_\zeta, f)\phi(k)dk\,\overline{W(\phi, \chi_\zeta)}d|\zeta|. \tag{99}$$

8.3 Concluding remarks. We have thus reached the goal we were aiming at, namely an absolutely convergent formula of the form (13). Recall f is a K-finite function which is itself a convolution product of two K-finite functions. We have denoted by K_G and K_{cusp} the geometric and cuspidal kernels determined by f and found for the difference

$$\int K_G(u, n)du\bar{\theta}(n)dn - \int K_{\mathrm{cusp}}(u, n)du\bar{\theta}(n)dn$$

the following expression:

$$\sum_\chi \sum_P \int \sum_{\phi\in\mathcal{B}_0(\pi)} \Xi(I_0(\pi\zeta, f)\phi, \pi_\zeta)\,\overline{W(\phi, \pi_\zeta)}\,d|\zeta|.$$

Here we sum over all cuspidal data χ. For each P we choose a suitable representative (P, M, π) in the class of χ. We sum only over those χ such that the induced representation $I_P(\pi)$ (which is itself induced from cuspidal representations in the terminology of [AC]) is Galois invariant. For such a χ, we denote by $\Xi(.,\pi_\zeta)$ an appropriate invariant linear form on the space of the induced representation $I_P(\pi_\zeta)$. The inner sum is over an orthonormal basis of the induced representation and the integral is over the space $i\mathfrak{a}^*$ or the one dimensional subspace defined by $\zeta_1+\zeta_2=0$. Finally, the above expression is absolutely convergent in the sense that

$$\sum_\chi \sum_P \int \left| \sum_{\phi\in\mathcal{B}_0(\pi)} \Xi(I_0(\pi\zeta,f)\phi,\pi_\zeta)\,\overline{W(\phi,\pi_\zeta)} \right| d|\zeta| < +\infty.$$

References

[A1] J. Arthur, *A trace formula for Reductive Groups I, Terms associated to classes in $G(\mathbb{Q})$*, Duke Mathematical Journal **45** (1978), 911–952.

[A2] J. Arthur, *A trace formula for Reductive Groups I: Applications of a truncation operator*, Compositio Mathematica **40** (1980), 87–121.

[A3] J. Arthur, *On a family of distributions obtained from Eisenstein series II: Explicit formulas*, American Journal of Mathematics **104** (1982), 1289–1336.

[AC] J. Arthur and L. Clozel, *Simple Algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, No **120**, Princeton University Press, 1989.

[AG] A. Ash and D. Ginzburg, *p-adic L-functions for* GL$(2n)$, preprint.

[yF1] Y. Flicker, *On distinguished representations*, Journal für die Reine und Angewandte Mathematik **418** (1991), 139–172.

[yF2] Y. Flicker, *Distinguished representations and a Fourier summation formula*, Bulletin de la Société Mathématique de France **120** (1992), 416–465.

[yF3] Y. Flicker, *A Fourier summation formula for the symmetric space* $\mathrm{GL}(n)/\mathrm{GL}(n-1)$, Compositio Mathematica **88** (1993), 39–117.

[FJ] S. Friedberg and H. Jacquet, *Linear periods*, Journal für die Reine und Angewandte Mathematik **443** (1993), 91–139.

[J1] H. Jacquet, *Sur un résultat de Waldspurger*, Annales Scientifiques de l'École Normale Supérieure **19** (1986), 185–229.

[J2] H. Jacquet, *Sur un résultat de Waldspurger II*, Compositio Mathematica **63** (1987), 315–389.

[J3] H. Jacquet, *On the non vanishing of some L-functions*, Indian Academy of Sciences. Proceedings. Mathematical Sciences **97** (1987), 117–155.

[J4] H. Jacquet, *Représentations distinguées pour le groupe orthogonal*, Comptes Rendus de l'Academie des Sciences, Paris **312** (1991), Série I, 957–961.

[J5] H. Jacquet, *Relative Kloosterman Integrals for* GL(3), *II*, Canadian Journal of Mathematics **44** (1992), 1220–1240.

[JR] H. Jacquet and S. Rallis, *Kloosterman Integrals for skew symmetric matrices*, Pacific Journal of Mathematics **154** (1992), 265–283.

[JY1] H. Jacquet and Y. Ye, *Une remarque sur le changement de base quadratique*, Comptes Rendus de l'Academie des Sciences, Paris **311** (1990), Série I, 671–676.

[JY2] H. Jacquet and Y. Ye, *Relative Kloosterman integrals for* GL(3), Bulletin de la Société Mathématique de France **120** (1992), 263–295.

[HLR] G. Harder, R. P. Langlands and M. Rapoport, *Algebraische Zyklen auf Hilbert-Blumenthal-Flachen*, Journal für die Reine und Angewandte Mathematik **366** (1986), 53–120.

[L] S. Lang, *Algebraic Number Theory*, Springer-Verlag, Berlin, 1986.

[M1] Z. Mao, *Relative Kloosterman integrals for the unitary group*, Comptes Rendus de l'Academie des Sciences, Paris **315 I** (1992), 381–386.

[M2] Z. Mao, *Sur les sommes de Salié relatives*, Comptes Rendus de l'Academie des Sciences, Paris **316 I** (1993), 1257–1262.

[M3] Z. Mao, *Relative Kloosterman integrals for* GL(3), to appear in the Canadian Journal of Mathematics.

[MW] C. Moeglin and J. L. Waldspurger, *Le spectre résiduel de* GL(n), Annales Scientifiques de l'École Normale Supérieure **22** (1989), 605–674.

[t.S] T. Springer, *Some results on algebraic groups with involution*, in *Algebraic Groups and Related Topics*, Advanced Studies in Pure Mathematics **6** (1983), 523–543.

[Y1] Y. Ye, *Kloosterman integrals and base change*, Journal für die Reine und Angewandte Mathematik **400** (1989), 57–121.

[Y2] Y. Ye, *The fundamental lemma of a relative trace formula for* GL(3), Compositio Mathematica **89** (1993), 121–162.

[Y3] Y. Ye, *Orbital integrals of a relative trace formula for* GL(3), Chinese Science Bulletin **38** (1993), 969–972.

[Y4] Y. Ye, *An integral transform and its applications*, preprint.